HWM–05–05
EMPG–05–06
hep–th/0504202

April 2005

Finite Volume Gauge Theory Partition Functions in Three Dimensions

Richard J. Szabo

Department of Mathematics
School of Mathematical and Computer Sciences
Heriot-Watt University
Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, U.K.
R.J.Szabo@ma.hw.ac.uk

Abstract

We determine the fermion mass dependence of Euclidean finite volume partition functions for three-dimensional QCD in the ϵ-regime directly from the effective field theory of the pseudo-Goldstone modes by using zero-dimensional non-linear σ-models. New results are given for an arbitrary number of flavours in all three cases of complex, pseudo-real and real fermions, extending some previous considerations based on random matrix theory. They are used to describe the microscopic spectral correlation functions and smallest eigenvalue distributions of the QCD_3 Dirac operator, as well as the corresponding massive spectral sum rules.

1 Introduction and Summary

The ϵ-regime of quantum chromodynamics in four spacetime dimensions (QCD_4) [1, 2] provides one of the few examples wherein exact nonperturbative results can be derived from QCD. It corresponds to the infrared sector of the Euclidean Dirac operator spectrum which is related to the mechanism of chiral symmetry breaking. The ϵ-expansion of chiral perturbation theory in this case, which describes the dynamics of the Goldstone modes [3], is taken with respect to the small quantity $\epsilon = L^{-1}$ where L is the linear size of the given spacetime volume. If L is much larger than the QCD scale, then spontaneously broken chiral symmetry dictates the interactions and the leading contributions come from the zero-momentum pseudo-Goldstone fields. This regime represents a domain wherein a detailed understanding can be achieved of the lowest-lying Dirac operator eigenvalues, which dominate the physical low-energy observables, and in which numerical computations in lattice QCD are possible [4]. Most of the recent understanding of the ϵ-regime of QCD comes from its relation to universal random matrix theory results [5, 6].

In this paper we will study the infrared behaviour of the Dirac operator spectrum in three-dimensional QCD (QCD_3). In this case the analog of chiral symmetry breaking is played by the spontaneous breaking of flavour symmetry [7]. The finite volume gauge theory is connected to three universality classes in random matrix theory, the unitary ensemble (fundamental fermions with $N_c \geq 3$ colours) [8, 9], the orthogonal ensemble (fundamental fermions with colour group $SU(2)$) [10] and the symplectic ensemble (adjoint fermions with colour group $SU(N_c)$) [11]. The random matrix theory representation of the effective QCD_3 partition function in the ϵ-regime has lead to a number of conjectures regarding the derivation of the spectral properties of the three-dimensional Dirac operator from joint eigenvalue probability distributions in the pertinent matrix model [8, 12]. The patterns of flavour symmetry breaking in all three instances have been identified [8, 10, 11] and supported by Monte Carlo simulations [13], and the microscopic spectral correlation functions have been calculated [6],[14]–[18].

To derive nonperturbative analytic results for the smallest eigenvalues of the Dirac operator it is necessary to have a complete proof that the universal random matrix theory results coincide with the low-energy limit of the effective field theory. From the field theoretic point of view [2] these observables should be computable entirely within the framework of finite volume partition functions which are generating functionals for the order parameter for flavour symmetry breaking [15]. For QCD_3 this problem has only been thoroughly investigated for the unitary ensemble by using the supersymmetric formulation of partially quenched effective Lagrangians [19], the replica limit [20], and the relationship [21] between finite volume partition functions and the τ-function of an underlying integrable KP hierarchy [22, 23]. The goal of this paper is to derive finite volume partition functions in all three cases mentioned above and to use them to explore spectral characteristics of the QCD_3 Dirac operator directly in the low-energy effective field theory.

In all three ensembles the effective field theory is described by a non-linear σ-model in a maximally symmetric compact space [8, 10, 11]. These target spaces fall into the classical Cartan classification of symmetric spaces, and confirm part of the Zirnbauer classification of random matrix universality classes in terms of Riemannian symmetric superspaces [24]. This enables us to exploit the well-known geometrical properties of these spaces [25, 26] (see [27] for a recent review in the context of random matrix theory) and hence explore the differences

between the three classes of fermions in QCD_3. For example, we will find that the finite volume partition functions for adjoint fermions are drastically different in functional form from those for complex and pseudo-real fermions, and this difference can be understood from the form of the Riemannian metrics on the corresponding symmetric spaces. A similar geometric feature also exhibits the difference between even and odd numbers of quark flavour components. In the latter cases the theory is plagued by a severe sign problem which is rather delicate from the effective field theory point of view [8]. We derive new results in all three instances for the finite volume partition functions with odd numbers of flavours. In the case of the unitary ensemble, we use them to derive a new simplified expression for the microscopic spectral correlators, extending the known effective field theory results for the even-flavoured case [19]. We also study some of the spectral sum rules which constrain the eigenvalues of the Dirac operator [8, 2, 28], and briefly describe its smallest eigenvalue distribution functions [29].

On a technical level, we present a complete and thorough analysis only for the case of complex fermions. In the cases of pseudo-real and real fermions, we complete the derivations only in the cases of completely degenerate mass eigenvalues, as there are some algebraic obstructions which prevent us from obtaining closed analytical expressions for the finite volume partition functions for arbitrary configurations of quark masses. Nevertheless, we present a very general algebraic description of the differences and similarities between the three universality classes, and also of the distinction between the even and odd flavoured cases, based on a remnant discrete subgroup of the flavour symmetry acting on the mass parameters of the gauge theory. This residual symmetry is described in general in Section 2, where we also review various aspects of the non-linear σ-models and describe the geometry of their symmetric target spaces in a general setting that applies also to other symmetric space σ-models such as those encountered in QCD_4. Sections 3–5 then use this general formalism to analyse in detail the three classes of fermions in turn. In addition to obtaining new results in the odd-flavoured cases, the main new predictions of the effective field theory approach concern the (equal mass) partition functions for adjoint fermions, which have a far more intricate functional structure than the other ensembles and which involve functions usually encountered in QCD_4 [2, 30], not in QCD_3.

2 Finite Volume Partition Functions

The Euclidean Dirac operator governing the minimal coupling of quarks in the QCD_3 action is given by

$$\mathrm{i}\,\not{D} = \mathrm{i}\,\sigma^\mu\,\left(\partial_\mu + \mathrm{i}\,A_\mu\right)\ , \tag{2.1}$$

where σ^μ, $\mu = 1, 2, 3$ are the Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ,\quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\ ,\quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ , \tag{2.2}$$

and A_μ are gauge fields of the local colour group $SU(N_c)$, $N_c \geq 2$. When multi-colour fermion fields are coupled to this operator, the three-dimensional field theory also possesses a global flavour symmetry described by rotations of the quark field components via elements of some Lie group G. If one introduces quark masses in equal and opposite pairs, then the fermion determinant is positive definite and the quantum field theory is anomaly free [8]. The addition

of such mass terms preserves three-dimensional spacetime parity [16] but breaks the flavour symmetry, and thereby permits one to analyse the possibility of spontaneous flavour symmetry breaking in QCD_3. In this section we will construct the general low-energy effective field theory of the associated pseudo-Goldstone bosons in the ergodic ϵ-regime wherein exact, nonperturbative information about the spectrum of the QCD_3 Dirac operator (2.1) can be obtained.

2.1 Non-Linear σ-Models of QCD_3

We are interested in the mechanism of flavour symmetry breaking in three-dimensional QCD, which we consider defined at fixed ultraviolet cutoff Λ, while the quantum field theory in the infrared is regulated by a three-volume V. The effective field theory for the low-momentum modes of the Nambu-Goldstone bosons can be derived by using the three-dimensional version of chiral perturbation theory [3]. It is consistent with the flavour symmetry of the original quantum field theory represented through transformations of the quark fields by the Lie group G. In the ergodic regime, the effective Lagrangian in Euclidean space is given by

$$L_{\rm eff} = \frac{f_\pi^2}{4}\ {\rm Tr}\left(\partial_\mu\mathcal{U}\,\partial^\mu\mathcal{U}^*\right) - \frac{\Sigma}{2}\ {\rm Tr}\left[\mathcal{M}\left(\mathcal{U}+\mathcal{U}^*\right)\right] + O\left(\mathcal{M}^2\right)\ , \tag{2.3}$$

where f_π is the decay constant of the pseudo-Goldstone modes, Σ is the infinite-volume quark-antiquark condensate which is the order parameter for flavour symmetry breaking, and $\mathcal{M}$ is the $N_f \times N_f$ mass matrix induced by integrating out the $N_f \geq 1$ flavours of fermion fields in the QCD_3 partition function. When the number of fermion flavours $N_f = 2n$ is even, so that n is the number of four-component spinors, we assume that the eigenvalues of $\mathcal{M}$ occur in parity-conjugate pairs $(m_i, -m_i)$, $m_i \geq 0$, $i = 1, \dots, n$. When $N_f = 2n+1$ is odd, there is a single unpaired quark of mass $m_0 \geq 0$ which breaks parity symmetry explicitly when $m_0 > 0$ and radiatively when $m_0 = 0$ [31]. The higher order terms in (2.3) represent contributions involving gluons and confined quarks. The symbol Tr, in the representations that we shall use in this paper, will always correspond to an ordinary $N_f \times N_f$ matrix trace. The Nambu-Goldstone field $\mathcal{U}$ lives in the appropriate vacuum manifold for the given symmetry breaking. We will use the Dyson index $\beta = 1, 2, 4$, which is borrowed from random matrix theory terminology [32]. It labels the anti-unitary symmetries of the QCD_3 Dirac operator which in turn depends on the types of fermions that are present in the original field theory [9].

The Goldstone manifold is a coset space

$$\mathcal{G}_\beta(N_f) \ = \ G\,/\,H_\Gamma\ , \tag{2.4}$$

where H_Γ is the stability subgroup of the original flavour symmetry group G which leaves fixed an $N_f \times N_f$ parity matrix Γ representing an involutive automorphism of G that reflects the parity symmetry of the original field theory. The fields in (2.3) may then be parametrized as

$$\mathcal{U} = U\,\Gamma\,U^{\rm t} \tag{2.5}$$

with $U \in \mathcal{G}_\beta(N_f)$. In (2.5) the superscript t denotes the appropriate conjugation involution on the Lie group G, given by $U^{\rm t} = U^\dagger$ for $\beta = 2$ and $U^{\rm t} = U^\top$ for $\beta = 1, 4$.

Given the non-linear σ-model (2.3), we would now like to study the region where the zero-momentum mode of U dominates. This is the ϵ-regime in which the linear dimension of the system is much smaller than the Compton wavelength of the Nambu-Goldstone bosons. The

spacetime integration over the effective Lagrangian (2.3) then produces an overall volume factor V, and the partition function of the non-linear σ-model simplifies to a finite-dimensional coset integral. The region of interest is obtained by simultaneously taking the thermodynamic limit $V \to \infty$ and the chiral limit $\mathcal{M} \to \mathbf{0}_{N_f}$ of massless quarks, with the rescaled masses $M = V\,\Sigma\,\mathcal{M}$ finite. The effective field theory of the Goldstone bosons is then the correct description of the dynamics below the scale set by the mass gap. Because of the tuning of the quark masses to zero, the higher-order corrections to this effective theory are exponentially small in the mass gap and come from integrating out the heavy states.

The quantum field theory is thus adequately described by the finite volume partition function [8, 10, 11]

$$
\begin{aligned}
Z_\beta^{(N_f)}(M) \;=\; \int\limits_{\mathcal{G}_\beta(N_f)} \mathrm{d}\nu_\beta(U)\,\Big[&\exp\ \mathrm{Re}\ \mathrm{Tr}\,\big(M\,U\,\Gamma\,U^{\mathrm{t}}\big) \\
&+\tfrac{1}{2}\,\big(1-(-1)^{q_{N_f}}\big)\,(-1)^{N_{\not{D}}}\ \exp\ \mathrm{Re}\ \mathrm{Tr}\,\big(-M\,U\,\Gamma\,U^{\mathrm{t}}\big)\Big]
\end{aligned}
\tag{2.6}
$$

where q_{N_f} is the even/odd congruence class of the flavour number N_f given by $q_{2n} = 0$, $q_{2n+1} = 1$, and $N_{\not{D}}$ is the number of Dirac operator eigenvalues (defined, for example, in a lattice regularization or in random matrix theory).The second line of (2.6) only contributes when there is an odd number of fermion flavours. It arises from the fact that, generally, the QCD$_3$ partition function changes by a phase $(-1)^{N_f\,N_{\not{D}}}$ under a parity transformation of the quark masses. When $N_f = 2n+1$ is odd, the fermion determinant is not positive definite and its sign determines two gauge inequivalent vacuum states which both have to be included in the low-energy limit of the QCD$_3$ partition function in the ϵ-regime [8]. We will always ignore irrelevant numerical constants (independent of M) in the evaluation of (2.6), as they will not contribute to any physical quantities and can be simply absorbed into the normalization of the partition function. The measure $\mathrm{d}\nu_\beta(U)$ is inherited from the invariant Haar measure for integration over the Lie group G. It will be constructed in the next subsection.

The low-energy effective partition function of the zero-dimensional σ-model (2.6) contains sufficient information to completely constrain the low-energy Dirac operator spectrum. For instance, by matching exact results from random matrix theory with the low-energy effective field theory (2.6), it is possible to express the microscopic limit of the spectral k-point correlation functions

$$
\varrho_\beta^{(N_f)}(\lambda_1,\dots,\lambda_k;\mathcal{M}) = \left\langle \prod_{l=1}^{k} \mathrm{Tr}\ \delta(\lambda_l - \mathrm{i}\,\not{D} + \mathrm{i}\,\mathcal{M}) \right\rangle_\beta \tag{2.7}
$$

of the QCD$_3$ Dirac operator (2.1) in terms of a ratio of finite volume partition functions, namely (2.6) and one which involves βk additional species of valence quarks of imaginary masses $\mathrm{i}\,\lambda_l$ [15, 22]. In (2.7) the average is taken over $SU(N_c)$ gauge field configurations weighted by the three-dimensional Yang-Mills action. Both (2.6) and the microscopic limit $\rho_2^{(N_f)}$ of (2.7) depend only on the eigenvalues $\pm\,\mu_i$, $i = 1,\dots,n$ (and μ_0 when N_f is odd) of the scaled mass matrix M.

For $\beta = 2, 4$ one explicitly has

$$
\begin{aligned}
\rho_\beta^{(N_f)}(\zeta_1,\ldots,\zeta_k;M) &= C_\beta^{(k)} \prod_{l=1}^{k} (\,\mathrm{i}\,\zeta_l+\mu_0)^{q_{N_f}} \prod_{i=1}^{n} (\zeta_l^2+\mu_i^2) \prod_{m<p} |\zeta_m^2-\zeta_p^2|^\beta \\
&\quad \times \frac{Z_\beta^{(N_f+\beta k)}(M,\overbrace{\mathrm{i}\,\zeta_1,\ldots,\mathrm{i}\,\zeta_1}^{\beta},\ldots,\overbrace{\mathrm{i}\,\zeta_k,\ldots,\mathrm{i}\,\zeta_k}^{\beta})}{Z_\beta^{(N_f)}(M)}
\end{aligned} \tag{2.8}
$$

where $\zeta_l = V\,\Sigma\,\lambda_l$ are the unfolded Dirac operator eigenvalues, and the proportionality constant $C_\beta^{(k)}$ is fixed by the matching condition between the confluent limit of the microscopic correlators and the macroscopic correlation functions at the spectral origin as

$$
\lim_{\zeta_1,\ldots,\zeta_k\to\infty} \rho_\beta^{(N_f)}(\zeta_1,\ldots,\zeta_k;M) = \frac{\varrho_\beta^{(N_f)}(\overbrace{0,\ldots,0}^{k};\mathcal{M})}{(V\,\Sigma)^k} = \left(\frac{1}{\pi}\right)^k \ . \tag{2.9}
$$

For $\beta=1,2,4$, the finite volume partition functions may be used to compute the normalized hole probability $E_\beta^{(N_f)}(\zeta;M)$, i.e. the probability that the interval $[-\zeta,\zeta]\subset\mathbb{R}$ is free of Dirac operator eigenvalues. One has the formula [29]

$$
E_\beta^{(N_f)}(\zeta;M) = \mathrm{e}^{-\frac14\zeta^2}\ \frac{Z_\beta^{(N_f)}\left(\sqrt{M^2+\zeta^2}\right)}{Z_\beta^{(N_f)}(M)} \ , \tag{2.10}
$$

from which the smallest eigenvalue distribution is given by

$$
P_\beta^{(N_f)}(\zeta;M) = -\frac{\partial E_\beta^{(N_f)}(\zeta;M)}{\partial\zeta} \ . \tag{2.11}
$$

Furthermore, by matching (2.6) with the formal expansion of the full QCD$_3$ partition function

$$
\mathcal{Z}_\beta^{(N_f)}(\mathcal{M}) = \left\langle \prod_{i=1}^{n} \det\left(\not{D}^2+m_i^2\right)^{p_\beta}\ \det\left(\not{D}+\mathrm{i}\,m_0\right)^{q_{N_f}\,p_\beta} \right\rangle_\beta \tag{2.12}
$$

with $p_1=p_2=1$ and $p_4=\frac12$, one can express the mass-dependent susceptibilities in terms of derivatives of the finite volume partition functions with respect to one or more quark mass eigenvalues [28]. This leads to a set of massive spectral sum rules once (2.6) is known explicitly. The simplest such sum rule expresses the mass-dependent condensate for the flavour symmetry breaking in the case of equal quark masses as

$$
\left\langle \sum_{\zeta>0} \frac{1}{\zeta^2+\mu^2} \right\rangle_\beta = \frac{1}{2N_f\,\mu}\,\frac{\partial \ln Z_\beta^{(N_f)}(\mu,\ldots,\mu)}{\partial\mu} \ . \tag{2.13}
$$

In the massless case $\mu\to0$, the left-hand side of (2.13) can be computed directly from the coset integral (2.6) [8]–[11] and is given as a function of the Dyson index β by

$$
\left\langle \sum_{\zeta>0} \frac{1}{\zeta^2} \right\rangle_\beta = \frac{N_f-q_{N_f}}{2(N_f-1)\left(\frac{2(N_f-q_{N_f})}{\beta}+1\right)} \tag{2.14}
$$

for even values of $N_{\not{D}}$. The formula (2.14) generalizes the one given in [10, 11, 17, 33] to include the case of odd N_f. For $\beta = 2$ it was first obtained in [8]. Note that for an odd number of Dirac operator eigenvalues, the finite volume partition function (2.6) is not positive and vanishes when the unpaired fermion is massless, i.e. $\mu_0 = 0$ [16]. It is therefore not suitable to describe physical quantities such as chiral spectral sum rules, while it can appear radiatively [15].

2.2 Parametrization of the Goldstone Manifold

We will now describe how to evaluate the non-linear σ-model partition functions (2.6). For this, we shall first review some elementary Lie theory that will be of use to us and which will fix some notation. We assume that the quark flavour symmetry is represented by a compact, connected, simple Lie group G, which we take to be realized by $N_f \times N_f$ invertible matrices U. Let $\mathfrak{t}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G, so that $T = \exp(\,\mathrm{i}\,\mathfrak{t}\,)$ is a maximal torus of G. Any element $X \in \mathfrak{g}$ can be taken to lie in the Cartan subalgebra $\mathfrak{t}$ without loss of generality, since this can always be achieved via rotations by elements of G if necessary.

After performing such a rotation in the given Cartan decomposition of G, the Jacobian for the change of Haar integration measure is given by a Weyl determinant $\Delta_G(X)$ of G which is determined through the product

$$\Delta_G(X) = \prod_{\alpha>0} (\alpha, X) \ , \tag{2.15}$$

where (α, X) are the positive roots of G evaluated on the Cartan element X. Its most useful form is obtained by choosing an orthonormal basis $\vec{e}_i$, $i = 1, \ldots, \dim T$ of vectors in weight space, $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$. Then the root vectors α can be identified with dual elements

$$\alpha^\vee = \sum_{i=1}^{\dim T} \alpha_i \, \vec{e}_i \ , \tag{2.16}$$

while

$$X = \sum_{i=1}^{\dim T} x_i \, H_i \tag{2.17}$$

where H_i are the generators of $\mathfrak{t}$ in the matrix basis corresponding to the orthonormal weights. By identifying (2.17) with the weight vector $x^\vee = \sum_i x_i \, \vec{e}_i$, the pairing in (2.15) may then be written as the inner product

$$(\alpha, X) = \alpha^\vee \cdot x^\vee = \sum_{i=1}^{\dim T} \alpha_i \, x_i \ . \tag{2.18}$$

With $U = \exp(\,\mathrm{i}\, X)$, the suitably normalized invariant Haar measure on the Lie group G is given by

$$[\mathrm{d}U] = \prod_{i=1}^{\dim T} \mathrm{d}x_i \ \prod_{\alpha>0} \sinh^2(\alpha, \ln U) \ . \tag{2.19}$$

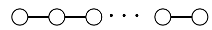
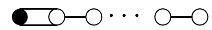
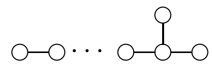
G	Positive Weights	H_i	Dynkin Diagram	Weyl Group
$U(r)$ $\beta{=}2$	$\vec{e}_i - \vec{e}_j$, $1 \leq i < j \leq r$	$E_{i,i}$, $1 \leq i \leq r$		S_r
$Sp(2r)$ $\beta{=}1$	$\vec{e}_i \pm \vec{e}_j$, $1 \leq i < j \leq r$ $2\vec{e}_i$, $1 \leq i \leq r$	$E_{i,i} - E_{i+r,i+r}$, $1 \leq i \leq r$		$S_r \ltimes (\mathbb{Z}_2)^r$
$SO(2r+1)$ $\beta{=}4$	$\vec{e}_i \pm \vec{e}_j$, $1 \leq i < j \leq r$ $\vec{e}_i$, $1 \leq i \leq r$	$E_{i,i} - E_{i+r,i+r}$, $1 \leq i \leq r$		$S_r \ltimes (\mathbb{Z}_2)^r$
$SO(2r)$ $\beta{=}4$	$\vec{e}_i \pm \vec{e}_j$, $1 \leq i < j \leq r$	$E_{i,i} - E_{i+r,i+r}$, $1 \leq i \leq r$		$S_r \ltimes (\mathbb{Z}_2)^{r-1}$

Table 1: *The flavour symmetry groups G of rank $r = \dim T \geq 2$ and their corresponding Dyson index β. The vectors $\vec{e}_i$, $i = 1, \dots, r$ form an orthonormal basis of weight space, while $E_{\mu,\nu}$ is a corresponding orthonormal basis of matrix units $(E_{\mu,\nu})_{\lambda\rho} = \delta_{\mu\lambda}\,\delta_{\nu\rho}$. The Dynkin diagrams all have r nodes. The symmetric group S_r permutes the eigenvalues x_i of a given Cartan element $X \in \mathfrak{t}$, while the cyclic group $\mathbb{Z}_2$ reflects them.*

Its infinitesimal form induces a measure on the Lie algebra $\mathfrak{g}$ as

$$[\mathrm{d}X] = \prod_{i=1}^{\dim T} \mathrm{d}x_i \ \Delta_G(X)^2 \ . \tag{2.20}$$

For the Lie groups that we shall deal with in this paper, the Cartan subalgebra $\mathfrak{t}$ is represented in the basis (2.17) by $N_f \times N_f$ diagonal matrices.

The residual flavour symmetry after such a transformation is given by the discrete Weyl group W_G of G which consists of the inequivalent transformations on $\mathfrak{t} \to \mathfrak{t}$ given by the adjoint actions

$$X \ \longmapsto \ X^w = w\,X\,w^{-1} \ , \quad w \in G \ . \tag{2.21}$$

Geometrically, W_G is the group of automorphisms of the root lattice of G corresponding to Weyl reflections. To each $w \in W_G$ we may associate a sign factor $\mathrm{sgn}(w) = \pm 1$ which is the even/odd $\mathbb{Z}_2$-parity of the Weyl element according to its action (2.21) on the diagonal elements of the Cartan matrices. For the partition functions that will be of interest to us in this paper, the relevant group theoretical data are summarized in Table 1.

We are now ready to construct the measure $\mathrm{d}\nu_\beta(U)$ for integration over the Goldstone manifold (2.4). For this, we shall introduce polar coordinates on the coset space $\mathcal{G}_\beta(N_f)$ [25], which in each case will be an irreducible, maximally symmetric compact space falling into the Cartan classification. Let

$$\mathfrak{g} = \mathfrak{h}_\Gamma \oplus \mathfrak{h}_\Gamma^\perp \tag{2.22}$$

be a Cartan decomposition of the simple algebra $\mathfrak{g}$ with respect to the stability subgroup $H_\Gamma = \exp(\,\mathrm{i}\,\mathfrak{h}_\Gamma)$, with $\mathfrak{h}_\Gamma^\perp = \mathfrak{g} \ominus \mathfrak{h}_\Gamma$ the orthogonal complement of $\mathfrak{h}_\Gamma$ in $\mathfrak{g}$ with respect to the Cartan-Killing form. The condition for (2.4) to be a symmetric space is then

$$[\mathfrak{h}_\Gamma\,,\,\mathfrak{h}_\Gamma] \subset \mathfrak{h}_\Gamma \ , \quad [\mathfrak{h}_\Gamma\,,\,\mathfrak{h}_\Gamma^\perp] \subset \mathfrak{h}_\Gamma^\perp \ , \quad [\mathfrak{h}_\Gamma^\perp\,,\,\mathfrak{h}_\Gamma^\perp] \subset \mathfrak{h}_\Gamma \ . \tag{2.23}$$

Let $\mathfrak{t}_\Gamma$ be a maximal abelian subalgebra in the subspace $\mathfrak{h}_\Gamma^\perp$, and let $\mathcal{N}_{\mathfrak{t}_\Gamma}$ be the normalizer subgroup of $\mathfrak{t}_\Gamma$ in H_Γ with $\mathcal{C}_{\mathfrak{t}_\Gamma} \subset \mathcal{N}_{\mathfrak{t}_\Gamma}$ its centralizer subgroup. As H_Γ is a compact symmetric subgroup of the flavour symmetry group G, every coset element $U \in \mathcal{G}_\beta(N_f)$ can be written by means of the adjoint representation of the stationary subgroup H_Γ as [25]

$$U = V\, R\, V^{-1} \ , \tag{2.24}$$

where $V \in H_\Gamma/\mathcal{C}_{\mathfrak{t}_\Gamma}$ while $R \in T_\Gamma = \exp(\mathrm{i}\, \mathfrak{t}_\Gamma)$ up to transformations by the adjoint actions of elements of the factor group $\mathcal{N}_{\mathfrak{t}_\Gamma}/\mathcal{C}_{\mathfrak{t}_\Gamma}$, which can be identified with the Weyl group of the restricted root system on the symmetric space $\mathcal{G}_\beta(N_f)$. This new root lattice differs from that inherited from the original group G if the Cartan subalgebra $\mathfrak{t}$ is a subset of $\mathfrak{h}_\Gamma$, and it is defined by completing the maximal abelian subalgebra $\mathfrak{t}_\Gamma \subset \mathfrak{h}_\Gamma^\perp$ with the generators in $\mathfrak{h}_\Gamma$ that commute with $\mathfrak{t}_\Gamma$ to give a new representation of $\mathfrak{t}$ that lies partly in $\mathfrak{h}_\Gamma^\perp$. The matrix R in (2.24) is the radial coordinate of the point $U \in \mathcal{G}_\beta(N_f)$, while V is its angular coordinate. This defines a foliation of the Goldstone manifold (2.4) by conjugacy classes under the adjoint action of the stability group H_Γ of the symmetric space.

The decomposition (2.24) means that every matrix $U \in \mathcal{G}_\beta(N_f)$ can be diagonalized by a similarity transformation in the subgroup H_Γ, and the radial coordinates $\{r_i\}$ are exactly the set of eigenvalues of U. Using the Haar measure (2.19) on the Lie group G, the corresponding Jacobian for the change of integration measure by (2.24) can be computed by standard techniques and one finds [26, 34]

$$\mathrm{d}\nu_\beta(U) = [\mathrm{d}V] \prod_{i=1}^{\dim T_\Gamma} \mathrm{d}r_i \prod_{\alpha_\Gamma>0} \Big|\sinh(\alpha_\Gamma, \ln R)\Big|^{m_{\alpha_\Gamma}} \ , \tag{2.25}$$

where $\ln R = \mathrm{i} \sum_i r_i\, H_i$ with $r_i \in [0, \frac{\pi}{2}]$ for $i = 1, \ldots, \dim T_\Gamma$. The second product in (2.25) goes over positive roots of the *restricted* root lattice on $\mathcal{G}_\beta(N_f)$, and m_{α_Γ} is the multiplicity of the root α_Γ in the given Cartan decomposition, i.e. the dimension of the subspace of raising operators corresponding to the root α_Γ in the algebra $\mathfrak{h}_\Gamma$ [26]. One can further exploit the ambiguity in the choice of radial coordinates R labelled by the restricted Weyl group $\mathcal{N}_{\mathfrak{t}_\Gamma}/\mathcal{C}_{\mathfrak{t}_\Gamma}$ to alternatively parametrize the radial decomposition (2.24) in a way more tailored to the evaluation of (2.6). The particular details of this parametrization will depend in general on the random matrix ensemble in which we are working. As we will see, the radial flavour symmetry group $\mathcal{N}_{\mathfrak{t}_\Gamma}/\mathcal{C}_{\mathfrak{t}_\Gamma}$ of the Goldstone manifold naturally distinguishes the cases $\beta = 1, 2$ from $\beta = 4$, and in all cases provides a nice algebraic distinction between the cases of an even or odd number of flavours N_f.

2.3 The Harish-Chandra Formula

The computation of (2.6) can be compared with the Harish-Chandra formula which computes the *group* integral [35]

$$\int_G [\mathrm{d}U]\ \exp\ \mathrm{Tr}\ \big(X\, U\, Y\, U^{-1}\big) = \frac{1}{\Delta_G(X)\, \Delta_G(Y)} \sum_{w\in W_G} \mathrm{sgn}(w)\ \exp\ \mathrm{Tr}\ \left(X\, Y^w\right) \tag{2.26}$$

for $X, Y \in \mathfrak{t}$. It expresses the group integral as a finite sum over terms, one for each element of the Weyl group, and it is intimately related to the Fourier transform of the Weyl character formula for the flavour symmetry group G. Its main characteristic is that the integration in

(2.26) may be rewritten as an integral over a coadjoint orbit, which is a symmetric space to which the Duistermaat-Heckman localization formula may be applied [36]. The Weyl group can be described as the factor group $W_G = \mathcal{N}_G/T$, where $\mathcal{N}_G$ is the normalizer subgroup of fixed points of the left action of T on the orbit G/T. In other words, the right-hand side of (2.26) is the stationary phase approximation to the group integral, and the Harish-Chandra formula states that in this case the approximation is *exact*. For this to be true it is important that the matrices X and Y live in the Lie algebra $\mathfrak{g}$ of G.

3 Fundamental Fermions

The simplest instance is when the quarks are described by complex fundamental Dirac fermion fields with $N_c \geq 3$ colours. Although these models have been the most extensively studied ones thus far, it will be instructive for later comparisons and simply for completeness to rederive the finite volume partition functions in this case using the formalism of the previous section. In matrix model language it corresponds to the unitary ensemble $\beta = 2$. The global flavour symmetry group is then $U(N_f)$, while the mass and parity matrices are given in a suitable basis of the vector space $\mathbb{C}^{N_f}$ by

$$
\begin{aligned}
M &= \begin{cases} \sigma^3 \otimes \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} , & N_f = 2n \ , \\ \begin{pmatrix} \mu_0 & \\ & \sigma^3 \otimes \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} \end{pmatrix} , & N_f = 2n+1 \ , \end{cases} \\
\Gamma &= \begin{cases} \sigma^3 \otimes 1\!\!1_n , & N_f = 2n \ , \\ \begin{pmatrix} 1 & \\ & \sigma^3 \otimes 1\!\!1_n \end{pmatrix} , & N_f = 2n+1 \ . \end{cases}
\end{aligned}
\tag{3.1}
$$

The mass matrix explicitly breaks the flavour symmetry, and its pairing into quark masses of opposite sign preserves three-dimensional spacetime parity. The Goldstone manifolds are [8]

$$
\mathcal{G}_2(2n) = U(2n) \, / \, U(n) \times U(n) \ , \tag{3.2}
$$

$$
\mathcal{G}_2(2n+1) = U(2n+1) \, / \, U(n+1) \times U(n) \tag{3.3}
$$

of real dimensions $\dim \mathcal{G}_2(2n) = 2n^2$ and $\dim \mathcal{G}_2(2n+1) = 2n\,(n+1)$.

3.1 Partition Functions

$N_f = 1$

The case of a single massive quark corresponds to formally setting $n = 0$ in (3.3). In this case there is no flavour symmetry breaking, the Goldstone manifold is a point, and there are no massless propagating degrees of freedom. The finite volume partition function is then trivially

obtained from (2.6) in the form

$$\boxed{Z_2^{(1)}(\mu) = \begin{cases} \cosh\mu & , \quad N_{\not D} \text{ even} \\ \sinh\mu & , \quad N_{\not D} \text{ odd} \end{cases}} \tag{3.4}$$

$N_f = 2n$

The restricted root lattice of the symmetric space (3.2) is the root lattice C_n of the symplectic group $Sp(2n)$ (see Table 1), with multiplicities $m_{\vec{e}_i \pm \vec{e}_j} = 2$, $m_{2\vec{e}_i} = 1$ for $i, j = 1, \ldots, n$, $i < j$ [25, 27]. The coset space measure (2.25) for the case $\beta = 2$ and an even number of fermion flavours is thereby given as

$$\mathrm{d}\nu_2(U) = [\mathrm{d}V] \prod_{i=1}^{n} \mathrm{d}r_i \ \sin 2r_i \ \prod_{i<j} \sin^2(r_i - r_j) \ \sin^2(r_i + r_j) \tag{3.5}$$

with $r_i \in [0, \frac{\pi}{2}]$. By using the trigonometric identity

$$2\sin(r_i - r_j) \ \sin(r_i + r_j) = \cos 2r_i - \cos 2r_j \tag{3.6}$$

and defining $\lambda_i = \cos 2r_i \in [-1, 1]$, we may write (3.5) in the simpler form

$$\mathrm{d}\nu_2(U) = \frac{[\mathrm{d}V]}{2^{n^2}} \prod_{i=1}^{n} \mathrm{d}\lambda_i \ \Delta(\lambda_1, \ldots, \lambda_n)^2 \ . \tag{3.7}$$

For any $X \in \mathfrak{u}(n)$ represented as a diagonal $n \times n$ matrix with entries $x_i \in \mathbb{R}$, we have introduced its Vandermonde determinant which is the Weyl determinant (2.15,2.18) for the unitary group $G = U(n)$ given by

$$\Delta_{U(n)}(X) = \prod_{i<j} (x_i - x_j) = \Delta(x_1, \ldots, x_n) \ . \tag{3.8}$$

In the polar decomposition (2.24), we define the $n \times n$ diagonal matrix

$$\rho = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix} \tag{3.9}$$

and write the radial coordinates as

$$R = \exp\left(\mathrm{i}\,\sigma^1 \otimes \rho\right) \tag{3.10}$$

which, since $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$, has the requisite property that it anticommutes with the parity matrix in (3.1) as

$$R\,\Gamma = \Gamma\,R^{-1} \ . \tag{3.11}$$

We decompose the angular degrees of freedom $V \in U(n) \times U(n)$ as

$$V = \begin{pmatrix} V_+ & \\ & V_- \end{pmatrix} \tag{3.12}$$

with $V_\pm \in U(n)$, so that they commute with the parity matrix as

$$V\,\Gamma = \Gamma\,V \ . \tag{3.13}$$

On substituting (2.24) into the finite volume partition function (2.6), and by using the identities (3.10)–(3.13) along with $\cos(\sigma^1 \otimes \eta) = \mathbb{1}_2 \otimes \cos\eta$ for any $\eta \in \mathfrak{u}(n)$, we may thereby write the action of the non-linear σ-model in this case as

$$\begin{aligned}
\mathrm{Re}\ \mathrm{Tr}\ \left(M\,U\,\Gamma\,U^\dagger\right) &= \tfrac{1}{2}\ \mathrm{Tr}\ \left[M\,\Gamma\,V\ \left(R^2 + R^{-2}\right)\ V^{-1}\right] \\
&= \mathrm{Tr}\ \left[M\,\Gamma\,V\ \cos\left(2\sigma^1 \otimes \rho\right)\ V^{-1}\right] \\
&= \mathrm{Tr}\ \left[\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} V_+ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} V_+^{-1}\right] \\
&\quad + \mathrm{Tr}\ \left[\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} V_- \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} V_-^{-1}\right] \ .
\end{aligned} \tag{3.14}$$

The resulting integrals over the two sets of angular coordinates $V_\pm \in U(n)$ are identical and can be evaluated by applying the Harish-Chandra formula (2.26) to the unitary group $G = U(n)$ (In this case (2.26) is often refered to as the Itzykson-Zuber formula [37] in the physics literature). The Weyl group in this case is the symmetric group $W_{U(n)} = S_n$ and it acts on X by permuting its eigenvalues, $(X^w)_i = (w\,X\,w^{-1})_i = x_{\hat w(i)}$, where $\hat w \in S_n$ is a permutation and $\mathrm{sgn}(w)$ is its parity. In this way the finite volume partition function (2.6) acquires the form

$$Z_2^{(2n)}(\mu_1,\dots,\mu_n) = \frac{1}{\Delta(\mu_1,\dots,\mu_n)^2}\sum_{\hat w_\pm \in S_n} \mathrm{sgn}\left(\hat w_+\,\hat w_-\right) \prod_{i=1}^{n}\ \int_{-1}^{1} \mathrm{d}\lambda_i\ \ \mathrm{e}^{\,\lambda_i(\mu_{\hat w_+(i)} + \mu_{\hat w_-(i)})} \tag{3.15}$$

involving the unfolded masses μ_i, $i = 1,\dots,n$. Since the integration measure and domain in (3.15) are invariant under permutations of the λ_i's, we may reduce the double sum over the Weyl group to a *single* sum over the relative permutation $\hat w = \hat w_+\,\hat w_-^{-1}$. The λ_i integrals are elementary and expressed in terms of the function

$$\mathsf{K}(x) = \frac{\sinh x}{x} = \frac{1}{2}\int_{-1}^{1} \mathrm{d}\lambda\ \ \mathrm{e}^{\,x\,\lambda} \tag{3.16}$$

which for purely imaginary argument defines the spectral sine-kernel of the unitary ensemble [32]. The sum over $\hat w \in S_n$ defines a determinant, and in this way one may finally arrive at the simple $n \times n$ determinant formula

$$\boxed{Z_2^{(2n)}(\mu_1,\dots,\mu_n) = \frac{\det\Big[\mathsf{K}(\mu_i + \mu_j)\Big]_{i,j=1,\dots,n}}{\Delta(\mu_1,\dots,\mu_n)^2}} \tag{3.17}$$

This is precisely the form of the finite volume partition function that was obtained in [19] from considerations based on partially-quenched effective field theories.

$\boldsymbol{N_f = 2n+1}$

The restricted root lattice of the symmetric space (3.3) is the non-reduced irreducible root lattice BC_n given by the union of the root systems B_n and C_n of the groups $SO(2n+1)$ and $Sp(2n)$ (see Table 1), with multiplicities $m_{\vec{e}_i \pm \vec{e}_j} = 2$, $m_{2\vec{e}_i} = 1$ as above, and $m_{\vec{e}_i} = 2$ [25, 27]. In this case the measure (2.25) thus reads

$$\begin{aligned} \mathrm{d}\nu_2(U) &= [\mathrm{d}V] \prod_{i=1}^n \mathrm{d}r_i \ \sin 2r_i \ \sin^2 r_i \ \prod_{i<j} \sin^2(r_i - r_j) \ \sin^2(r_i + r_j) \\ &= \frac{[\mathrm{d}V]}{2^{n\,(n+1)}} \prod_{i=1}^n \mathrm{d}\lambda_i \ (1-\lambda_i) \ \Delta(\lambda_1, \dots, \lambda_n)^2 \ , \end{aligned} \tag{3.18}$$

where we have used the double angle identity $\sin^2 r_i = \frac{1}{2}\,(1 - \cos 2r_i)$. With the definition (3.9), the radial coordinate matrix

$$R = \begin{pmatrix} 1 & \\ & \exp\left(\mathrm{i}\,\sigma^1 \otimes \rho\right) \end{pmatrix} \tag{3.19}$$

satisfies the requisite condition (3.11). The angular matrices $V \in U(n+1) \times U(n)$ are decomposed as in (3.12) with $V_+ \in U(n+1)$ and $V_- \in U(n)$, and the non-linear σ-model action is now modified to

$$\begin{aligned} \mathrm{Re}\ \mathrm{Tr}\left(M\,U\,\Gamma\,U^\dagger\right) &= \mathrm{Tr}\left[\begin{pmatrix} \mu_0 & & & \\ & \mu_1 & & \\ & & \ddots & \\ & & & \mu_n \end{pmatrix} V_+ \begin{pmatrix} 1 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} V_+^{-1}\right] \\ &\quad + \mathrm{Tr}\left[\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} V_- \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} V_-^{-1}\right] \ . \end{aligned} \tag{3.20}$$

The Harish-Chandra formula applied to the integrations over $V_+ \in U(n+1)$ and $V_- \in U(n)$ then yields

$$\begin{aligned} Z_2^{(2n+1)}(\mu_0, \mu_1, \dots, \mu_n) &= \frac{\prod\limits_{i=1}^n (\mu_i - \mu_0)^{-1}}{\Delta(\mu_1, \dots, \mu_n)^2} \sum_{\hat{w}_+ \in S_{n+1}} \ \sum_{\hat{w}_- \in S_n} \mathrm{sgn}\,(\hat{w}_+) \ \mathrm{sgn}\,(\hat{w}_-) \ \ \mathrm{e}^{\,\mu_{\hat{w}_+(0)}} \\ &\quad \times \prod_{i=1}^n \ \int\limits_{-1}^1 \mathrm{d}\lambda_i \ \ \mathrm{e}^{\,\lambda_i(\mu_{\hat{w}_+(i)} + \mu_{\hat{w}_-(i)})} \ + \ (-1)^{N_{\not{p}}} \ \left\{\begin{smallmatrix} \mu_0 \to -\mu_0 \\ \mu_i \to -\mu_i \end{smallmatrix}\right\} \ . \end{aligned} \tag{3.21}$$

Note how the extra factors of $(1-\lambda_i)$ in (3.18) have nicely cancelled against the $(n+1) \times (n+1)$ Vandermonde determinant $\Delta(1, \lambda_1, \dots, \lambda_n)$ coming from the V_+ integral. In the sum over $\hat{w}_- \in S_n$, we may regard S_n as the subgroup of the permutation group S_{n+1} acting only on the indices $i = 1, \dots, n$ in $\{0, 1, \dots, n\}$. Then as before we can truncate the double Weyl group sum in (3.21) to a single sum over $\hat{w} = \hat{w}_+\,\hat{w}_-^{-1} \in S_{n+1}$ by extending $\hat{w}_-$ to all of S_{n+1} through the definition $\hat{w}_-(0) = 0$. After performing the λ_i integrals, the finite volume partition function is

thus given by the simple $(n+1)\times(n+1)$ determinant formula

$$\boxed{\begin{aligned} Z_2^{(2n+1)}(\mu_0,\mu_1,\dots,\mu_n) = \frac{\prod_{i=1}^{n}(\mu_i-\mu_0)^{-1}}{\Delta(\mu_1,\dots,\mu_n)^2}\Bigg(&\det\begin{bmatrix} \mathrm{e}^{\,\mu_0} & \mathrm{e}^{\,\mu_j} \\ \mathsf{K}(\mu_0+\mu_i) & \mathsf{K}(\mu_i+\mu_j)\end{bmatrix}_{i,j=1,\dots,n} \\ &+(-1)^{n+N_{\not{D}}}\,\det\begin{bmatrix} \mathrm{e}^{\,-\mu_0} & \mathrm{e}^{\,-\mu_j} \\ \mathsf{K}(\mu_0+\mu_i) & \mathsf{K}(\mu_i+\mu_j)\end{bmatrix}_{i,j=1,\dots,n}\Bigg)\end{aligned}} \tag{3.22}$$

This is a new simplified expression for the finite volume partition function of QCD_3 with an odd number of fundamental fermions. It can be readily checked to coincide with previous expressions (given by $(2n+1)\times(2n+1)$ determinant formulas [15, 20]) for low numbers of flavours.

3.2 Correlation Functions

For $\beta=2$, the k-level correlation functions can be expressed as a ratio of partition functions with N_f and N_f+2k flavours as in (2.8). By using (3.17) one finds the scaled correlators for an even number of quark flavours given by

$$\boxed{\rho_2^{(2n)}(\zeta_1,\dots,\zeta_k;\mu_1,\dots,\mu_n) = \left(\frac{1}{\pi}\right)^k \frac{\det\begin{bmatrix} \mathsf{K}(\mu_i+\mu_j) & \mathsf{K}(\mu_i+\mathrm{i}\,\zeta_{l'}) \\ \mathsf{K}(\mu_j-\mathrm{i}\,\zeta_l) & \mathsf{K}(\mathrm{i}\,\zeta_l-\mathrm{i}\,\zeta_{l'})\end{bmatrix}_{\substack{i,j=1,\dots,n\\ l,l'=1,\dots,k}}}{\det\Big[\mathsf{K}(\mu_i+\mu_j)\Big]_{i,j=1,\dots,n}}} \tag{3.23}$$

where the matching condition (2.9) in this case fixes $C_2^{(k)}=(\frac{1}{2\pi})^k$. This again agrees with the field theoretic results of [19]. In the form (3.17), the expression (2.8), originally derived in the context of random matrix theory [15], is thereby naturally tied to the field theory derivation of the spectral functions $\rho_2^{(2n)}$.

For an odd number of flavours one finds

$$\boxed{\begin{aligned}&\rho_2^{(2n+1)}(\zeta_1,\dots,\zeta_k;\mu_0,\mu_1,\dots,\mu_n) \\ &= \frac{\det\begin{bmatrix} \mathrm{e}^{\,\mu_0} & \mathrm{e}^{\,\mu_j} & \mathrm{e}^{\,\mathrm{i}\,\zeta_{l'}} \\ \mathsf{K}(\mu_0+\mu_i) & \mathsf{K}(\mu_i+\mu_j) & \mathsf{K}(\mu_i+\mathrm{i}\,\zeta_{l'}) \\ \mathsf{K}(\mu_0-\mathrm{i}\,\zeta_l) & \mathsf{K}(\mu_j-\mathrm{i}\,\zeta_l) & \mathsf{K}(\mathrm{i}\,\zeta_l-\mathrm{i}\,\zeta_{l'})\end{bmatrix}_{\substack{i,j=1,\dots,n\\ l,l'=1,\dots,k}} + (-1)^{k+n+N_{\not{D}}}\left\{\begin{matrix}\mu_0\to-\mu_0\\ \mu_i\to-\mu_i\\ \zeta_l\to-\zeta_l\end{matrix}\right\}}{(2\pi)^k\,\det\begin{bmatrix} \mathrm{e}^{\,\mu_0} & \mathrm{e}^{\,\mu_j} \\ \mathsf{K}(\mu_0+\mu_i) & \mathsf{K}(\mu_i+\mu_j)\end{bmatrix}_{i,j=1,\dots,n} + (-1)^{n+N_{\not{D}}}\left\{\begin{matrix}\mu_0\to-\mu_0\\ \mu_i\to-\mu_i\end{matrix}\right\}}\end{aligned}} \tag{3.24}$$

This is a new general expression for the spectral k-point functions in the case of an odd number of quarks at $\beta=2$. It coincides with the known expressions from random matrix theory [16] and the replica method approach to QCD_3 [21]. In particular, the formula (3.24) shows explicitly that the correlators are not even functions of the scaled Dirac operator eigenvalues ζ_l for $M\neq\mathbf{0}_{2n+1}$ [21]. This expression also covers the case of a single flavour $N_f=1$ ($n=0$), which can be obtained by omitting the corresponding $n\times n$ blocks in the determinants above.

3.3 Sum Rules

We may work out the massive spectral sum rules (2.13) by taking the equal mass limits in (3.17) and (3.22). This yields indeterminate forms which can be computed by regulating the mass matrices in any way that removes the n-fold and $(n+1)$-fold eigenvalue degeneracies, and then taking the degenerate limits using l'Hôpital's rule. However, this is rather cumbersome to do in practise and it is much simpler to go back and work directly with the radial coset representation of the finite volume partition functions.

$\boldsymbol{N_f = 2n}$

From (3.14) we see that in the case of equal quark masses $\mu_i = \mu \ \ \forall i = 1, \ldots, n$, the σ-model action is independent of the angular degrees of freedom $V_\pm \in U(n)$ and the low-energy dynamics is mediated entirely by the n radial Goldstone boson degrees of freedom. The partition function (2.6) is then simply given by

$$Z_2^{(2n)}(\mu, \ldots, \mu) = \prod_{i=1}^{n} \int_{-1}^{1} \mathrm{d}\lambda_i \ \mathrm{e}^{\,2\mu\,\lambda_i} \ \Delta(\lambda_1, \ldots, \lambda_n)^2 \ . \tag{3.25}$$

We now use the identity [32]

$$\Delta(\lambda_1, \ldots, \lambda_n) = \det\left[\lambda_i^{j-1}\right]_{i,j=1,\ldots,n} \tag{3.26}$$

and expand the resulting two determinants in (3.25) into sums over permutations $\hat{w}_\pm \in S_n$. As in Section 3.1, the partition function depends only on the relative permutation $\hat{w} = \hat{w}_+ \, \hat{w}_-^{-1} \in S_n$, and the λ_i integrals are now expressed in terms of derivatives of the spectral kernel defined by (3.16) as

$$\mathsf{K}^{(m)}(x) = \frac{\mathrm{d}^m \mathsf{K}(x)}{\mathrm{d}x^m} = \frac{1}{2} \int_{-1}^{1} \mathrm{d}\lambda \ \lambda^m \ \mathrm{e}^{\,x\,\lambda} \ . \tag{3.27}$$

In this way we arrive at the $n \times n$ determinant formula

$$Z_2^{(2n)}(\mu, \ldots, \mu) = \det\Big[\mathsf{K}^{(i+j-2)}(2\mu)\Big]_{i,j=1,\ldots,n} \ . \tag{3.28}$$

We may present the partition function (3.28) in a somewhat more explicit form by expressing the generic derivatives (3.27) in terms of the sine-kernel $\mathsf{K}(x)$ itself. Starting from the integral representation (3.27) we write

$$\begin{aligned} \mathsf{K}^{(m)}(x) &= \frac{1}{2} \int_{-1}^{1} \mathrm{d}\lambda \ \mathrm{e}^{\,x\,\lambda} \left(1 - m \int_{\lambda}^{1} \mathrm{d}s \ s^{m-1}\right) \\ &= \mathsf{K}(x) - \frac{m}{2x} \int_{-1}^{1} \mathrm{d}s \ s^{m-1} \ \left(\mathrm{e}^{\,x\,s} - \mathrm{e}^{-x}\right) \ . \end{aligned} \tag{3.29}$$

This yields a simple linear inhomogeneous recursion relation for the sequence of functions $\mathsf{K}^{(m)}(x)$, $m \geq 0$ in (3.27) given by

$$\mathsf{K}^{(m)}(x) + \frac{m}{x}\,\mathsf{K}^{(m-1)}(x) = \mathsf{K}(x) + \frac{1}{2}\Big(1-(-1)^m\Big)\left(\frac{\cosh x}{x} - \mathsf{K}(x)\right) \ . \tag{3.30}$$

It is straightforward to iterate the recurrence (3.30) in m and express the solution in terms of $\mathsf{K}(x)$ and other elementary functions of x, and in this way the equal mass partition function (3.28) can be written as

$$\boxed{Z_2^{(2n)}(\mu,\dots,\mu) = \det\Big[\boldsymbol{K}_{i+j-2}(2\mu)\Big]_{i,j=1,\dots,n}} \tag{3.31}$$

where

$$\begin{aligned}\boldsymbol{K}_m(x) \ &= \ \frac{1}{2}\sum_{k=0}^m \frac{m!}{(m-k)!}\left(-\frac{1}{x}\right)^{k+1} \\ &\quad \times\Big[\Big(1-(-1)^{m+k}\Big)\Big(\sinh x - \cosh x\Big) - 2\sinh x\Big] \ . \end{aligned} \tag{3.32}$$

From (3.31) one may now proceed to derive the massive spectral sum rules (2.13). For the first few even numbers of quark flavours we find

$$\boxed{\begin{aligned}\left.\left\langle\sum_{\zeta>0}\frac{1}{\zeta^2+\mu^2}\right\rangle_2\right|_{N_f=2} &= \frac{2\mu\coth 2\mu - 1}{4\mu^2} \\ \left.\left\langle\sum_{\zeta>0}\frac{1}{\zeta^2+\mu^2}\right\rangle_2\right|_{N_f=4} &= \frac{1}{4\mu^2}\,\frac{2\sinh^2 2\mu - \mu\,\sinh 4\mu - 4\mu^2}{4\mu^2 - \sinh^2 2\mu} \\ \left.\left\langle\sum_{\zeta>0}\frac{1}{\zeta^2+\mu^2}\right\rangle_2\right|_{N_f=6} &= \frac{1}{12\mu^2}\,\frac{2\mu\big(16\mu^4+60\mu^2+3\big)\cosh 2\mu - 28\mu^2\big(4\mu^2+3\big)\sinh 2\mu - 6\mu\,\cosh^3 2\mu + 9\sinh^3 2\mu}{4\mu^2(4\mu^2+3)\sinh 2\mu - 16\mu^3\,\cosh 2\mu - \sinh^3 2\mu}\end{aligned}} \tag{3.33}$$

These all agree with (2.14) in the limit $\mu \to 0$. The first two sum rules in (3.33) were also obtained in [14].

$N_f = 2n+1$

From (3.18) and (3.20) it follows that in the equal mass limit $\mu_0 = \mu_i = \mu \ \ \forall i = 1,\dots,n$ the finite volume partition function (2.6) is given by

$$Z_2^{(2n+1)}(\mu,\dots,\mu) = \mathrm{e}^{\,\mu}\,\prod_{i=1}^n\,\int\limits_{-1}^{1}\mathrm{d}\lambda_i\ (1-\lambda_i)\ \mathrm{e}^{\,2\mu\,\lambda_i}\ \Delta(\lambda_1,\dots,\lambda_n)^2 \ \ + \ \ (-1)^{N_{\not{D}}}\ \{\mu\to-\mu\} \ . \tag{3.34}$$

Proceeding exactly as above, and using the fact that the functions (3.32) obey the reflection property $\boldsymbol{K}_m(-x) = (-1)^m\, \boldsymbol{K}_m(x)$, we can write (3.34) explicitly as the $n \times n$ determinant formula

$$\boxed{\begin{aligned} Z_2^{(2n+1)}(\mu,\dots,\mu) = \ & \mathrm{e}^{\,\mu}\ \det\Big[\boldsymbol{K}_{i+j-2}(2\mu) - \boldsymbol{K}_{i+j-1}(2\mu)\Big]_{i,j=1,\dots,n} \\ & + (-1)^{N_{\not{D}}}\ \mathrm{e}^{\,-\mu}\ \det\Big[\boldsymbol{K}_{i+j-2}(2\mu) + \boldsymbol{K}_{i+j-1}(2\mu)\Big]_{i,j=1,\dots,n} \end{aligned}} \tag{3.35}$$

with the determinants omitted for the $n = 0$ case as before. For an even number $N_{\not{D}}$ of Dirac operator eigenvalues, from (3.4) and (3.35) we obtain the massive spectral sum rules

$$\boxed{\begin{aligned} \left.\left\langle \sum_{\zeta>0} \frac{1}{\zeta^2+\mu^2} \right\rangle_2 \right|_{N_f=1} &= \frac{\tanh\mu}{2\mu} \\ \left.\left\langle \sum_{\zeta>0} \frac{1}{\zeta^2+\mu^2} \right\rangle_2 \right|_{N_f=3} &= \frac{1}{6\mu^2}\ \frac{2\mu^2\ \coth\mu - \mu(1+3\cosh 2\mu) - 2\sinh 2\mu}{2\mu + \sinh 2\mu} \end{aligned}} \tag{3.36}$$

which again agree with (2.14) in the limit $\mu \to 0$.

4 Pseudo-Real Fermions

The next simplest case is the parity-invariant Dirac operator $\mathrm{i}\,\not{D}$ acting on $N_c = 2$ colours of Dirac quarks in the fundamental representation of the $SU(N_c)$ gauge group. In contrast to the previous case, the Dirac operator now possesses a special anti-unitary symmetry, $[\mathfrak{C}, \mathrm{i}\,\not{D}] = 0$, generated by an operator $\mathfrak{C}$ with $\mathfrak{C}^2 = 1\!\!1$ [9, 10]. The operator $\mathfrak{C}$ can be expressed as a combination of the usual charge conjugation $C = \mathrm{i}\,\sigma^2$ with $C\,(\sigma^\mu)^*\,C^{-1} = \sigma^\mu$ for $\mu = 1, 2, 3$, an infinitesimal rotation by the matrix $\mathrm{i}\,\sigma^2$ in colour space, and complex conjugation. In a basis of spinors $|\psi_k\rangle$ wherein $\mathfrak{C}|\psi_k\rangle = |\psi_k\rangle$, the matrix elements $\langle\psi_k|\,\mathrm{i}\,\not{D}\,|\psi_l\rangle$ are all real [10]. This basis is obtained by augmenting the flavour components of the fermion fields by the two-colour group, thereby enhancing the original $U(N_f)$ flavour symmetry to a higher pseudo-real global flavour symmetry group $Sp(2N_f)$. In matrix model language this corresponds to the orthogonal ensemble $\beta = 1$.

The group $USp(2n)$ acts on the vector space $\mathbb{C}^{2n}$ preserving its canonical inner product as well as a symplectic structure $\mathbb{J}_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, i.e. a real antisymmetric $2n \times 2n$ matrix with $\mathbb{J}_n^2 = -1\!\!1_{2n}$. In a suitable basis of $\mathbb{C}^{2n}$, it consists of unitary matrices V satisfying

$$V\,\mathbb{J}_n\,V^\top = \mathbb{J}_n \ , \quad \mathbb{J}_n = \mathrm{i}\,\sigma^2 \otimes 1\!\!1_n \ . \tag{4.1}$$

With respect to the same basis, the mass and parity matrices are given by

$$
M = \begin{cases} \sigma^3 \otimes \mathrm{i}\,\sigma^2 \otimes \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} & , \quad N_f = 2n \ , \\ \begin{pmatrix} \mu_0 \ \mathrm{i}\,\sigma^2 & \\ & \sigma^3 \otimes \mathrm{i}\,\sigma^2 \otimes \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} \end{pmatrix} & , \quad N_f = 2n+1 \ , \end{cases}
$$

$$
\Gamma = \begin{cases} \sigma^3 \otimes \mathbb{J}_n & , \quad N_f = 2n \ , \\ \begin{pmatrix} \mathrm{i}\,\sigma^2 & \\ & \sigma^3 \otimes \mathbb{J}_n \end{pmatrix} & , \quad N_f = 2n+1 \ . \end{cases} \tag{4.2}
$$

The antisymmetric forms of these matrices follow from the Grassmann nature of the quark spinor fields ψ_k described above. Again the mass matrices break the $USp(4n)$ and $USp(4n+2)$ flavour symmetries, and the corresponding Goldstone manifolds are

$$
\mathcal{G}_1(2n) = USp(4n) \,/\, USp(2n) \times USp(2n) \ , \tag{4.3}
$$

$$
\mathcal{G}_1(2n+1) = USp(4n+2) \,/\, USp(2n+2) \times USp(2n) \tag{4.4}
$$

of real dimensions $\dim \mathcal{G}_1(2n) = 4n^2$ and $\dim \mathcal{G}_1(2n+1) = 4n\,(n+1)$. As expected, the number of Goldstone bosons here is exactly twice that of the previous section.

4.1 Partition Functions

$N_f = 2n$

The restricted root lattice of the symmetric space (4.3) is again the root space C_n of $Sp(2n)$, but now with multiplicities $m_{\vec{e}_i \pm \vec{e}_j} = 4$, $m_{2\vec{e}_i} = 3$ [25, 27]. The corresponding radial integration measure (2.25) on $\mathcal{G}_1(2n)$ is thus

$$
\begin{aligned}
\mathrm{d}\nu_1(U) &= [\mathrm{d}V] \prod_{i=1}^{n} \mathrm{d}r_i \ \sin^3 2r_i \ \prod_{i<j} \sin^4(r_i - r_j) \ \sin^4(r_i + r_j) \\
&= \frac{[\mathrm{d}V]}{2^{n\,(2n-1)}} \ \prod_{i=1}^{n} \mathrm{d}\lambda_i \ \left(1 - \lambda_i^2\right) \ \Delta(\lambda_1, \dots, \lambda_n)^4 \ ,
\end{aligned} \tag{4.5}
$$

where we have again defined $\lambda_i = \cos 2r_i \in [-1, 1]$. With ρ defined as in (3.9), we parametrize the radial element R as

$$
R = \exp\left(\mathrm{i}\,\sigma^1 \otimes \xi\right) \tag{4.6}
$$

where the matrix

$$
\xi = \sigma^2 \otimes \rho \tag{4.7}
$$

commutes with the symplectic structure as

$$
\xi\, \mathbb{J}_n = \mathbb{J}_n\, \xi \ . \tag{4.8}
$$

It follows that $R^\top = R^{-1} = R^\dagger$, and the radial matrix (4.6) anticommutes with the parity matrix Γ as in (3.11). For the angular degrees of freedom $V \in USp(2n) \times USp(2n)$ we again take the decomposition (3.12) with $V_\pm \in USp(2n)$, so that from (4.1) we have

$$V\,\Gamma\,V^\top = \Gamma \ . \tag{4.9}$$

Using these relations, along with the identity $\cos(\sigma^2 \otimes \eta) = \mathbb{1}_2 \otimes \cos\eta$ for any $\eta \in \mathfrak{u}(n)$, we can now write the action of the non-linear σ-model (2.6) similarly to (3.14) as

$$\begin{aligned}
\mathrm{Re}\ \mathrm{Tr}\left(M\,U\,\Gamma\,U^\top\right) &= \tfrac{1}{2}\ \mathrm{Tr}\left[\Gamma\,M\,V\ \left(R^2+R^{-2}\right)\,V^{-1}\right] \\
&= \mathrm{Tr}\left[\Gamma\,M\,V\,(\mathbb{1}_4 \otimes \cos 2\rho)\,V^{-1}\right] \\
&= \mathrm{Tr}\left[\left(\mathbb{1}_2 \otimes \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}\right) V_+ \left(\mathbb{1}_2 \otimes \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\right) V_+^{-1}\right] \\
&\quad + \mathrm{Tr}\left[\left(\mathbb{1}_2 \otimes \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}\right) V_- \left(\mathbb{1}_2 \otimes \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\right) V_-^{-1}\right] .
\end{aligned} \tag{4.10}$$

The finite volume partition function (2.6) thereby becomes

$$Z_1^{(2n)}(\mu_1,\dots,\mu_n) = \prod_{i=1}^n\ \int_{-1}^{1} \mathrm{d}\lambda_i\ \left(1-\lambda_i^2\right)\ \Delta(\lambda_1,\dots,\lambda_n)^4\ \Omega_1^{(n)}(\lambda_1,\dots,\lambda_n;\mu_1,\dots,\mu_n)^2 \tag{4.11}$$

where

$$\begin{aligned}
&\Omega_1^{(n)}(\lambda_1,\dots,\lambda_n;\mu_1,\dots,\mu_n) \\
&\quad = \int\limits_{Sp(2n)} [\mathrm{d}V]\ \exp\ \mathrm{Tr}\left[\left(\mathbb{1}_2 \otimes \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}\right) V \left(\mathbb{1}_2 \otimes \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\right) V^{-1}\right] .
\end{aligned} \tag{4.12}$$

It is not at all clear at this stage how to proceed with the calculation. The Lie algebra $\mathfrak{sp}(2n)$ consists of $2n \times 2n$ matrices X obeying the equation $X\,\mathbb{J}_n + \mathbb{J}_n\,X^\top = 0$. The diagonal source matrices in (4.12) are *not* elements of this Lie algebra, nor are the original mass and parity matrices in (4.2) elements of $\mathfrak{sp}(4n)$. Thus the Harish-Chandra formula (2.26) cannot be immediately applied to the angular integral (4.12). For the same reason this integral cannot be evaluated by using character expansion techniques for the symplectic group $Sp(2n)$ [38]. It would be interesting to determine the corresponding analytical continuation of the Harish-Chandra formula to this case, in analogy with that done in the unitary case $G = U(n)$ [39] which is based on the fact that the representation theories of the Lie groups $U(n)$ and $GL(n,\mathbb{C})$ are essentially the same. Note that the integral $\Omega_1^{(1)}(\lambda;\mu)$ can be computed by means of the Harish-Chandra formula for $SU(2) \cong Sp(2)$.

In [18] the finite volume partition function in this case is evaluated by applying a formalism based on skew-orthogonal polynomials in the orthogonal ensemble of random matrix theory, resulting in the Pfaffian expression

$$Z_1^{(2n)}(\mu_1,\dots,\mu_n) = \frac{\mathrm{Pfaff}\Big[\mathsf{K}'\left(\tilde\mu_a-\tilde\mu_b\right)\Big]_{a,b=1,\dots,2n}}{\prod\limits_{i=1}^n \mu_i\ \Delta\left(\mu_1^2,\dots,\mu_n^2\right)^2} \tag{4.13}$$

where $\tilde\mu_{2i-1}=-\mu_i$, $\tilde\mu_{2i}=\mu_i$ for $i=1,\dots,n$. Using the permutation expansion of the Pfaffian, the integral representation (3.27) and by comparing (4.13) with (4.11), it is possible to write a conjectural formula for the integral (4.12). There are several qualitative features of the expression (4.13) that make its derivation directly in field theory appear feasible. First of all, the denominator of (4.13) can be expressed in terms of the square of the Weyl determinant $\Delta_{Sp(2n)}(X)$ of an element $X\in\mathfrak{sp}(2n)$ with eigenvalues $\mu_1,\dots,\mu_n$ (see (2.15) and Table 1), as appears in the expression (2.26). Furthermore, the Weyl group $W_{Sp(2n)}$ is composed of permutations of eigenvalues $x_i\mapsto x_{\pi(i)}$, $i=1,\dots n$, $\pi\in S_n$, along with reflections $x_i\mapsto -x_i$. After the appropriate radial integration, the sum over $W_{Sp(2n)}$ can thereby produce the required Pfaffian in the numerator of the expression (4.13), which in the random matrix theory calculation arises naturally from quaternion determinants generated in the skew-orthogonal polynomial formalism. In the next subsection shall explicitly work out the σ-model partition functions in the case of equal quark masses and thus provide further evidence that the direct field theory calculation indeed does reproduce this Pfaffian expression.

$N_f = 2n+1$

The restricted root lattice of the symmetric space (4.4) is again the root space BC_n, with multiplicities $m_{\vec e_i\pm\vec e_j}=4$, $m_{2\vec e_i}=3$ as above, and in addition $m_{\vec e_i}=4$ [25, 27]. The coset space measure (2.25) in this case is thus given by

$$\begin{aligned}\mathrm{d}\nu_1(U) &= [\mathrm{d}V]\ \prod_{i=1}^n \mathrm{d}r_i\ \sin^3 2r_i\ \sin^4 r_i\ \prod_{i<j}\sin^4(r_i-r_j)\ \sin^4(r_i+r_j)\\ &= \frac{[\mathrm{d}V]}{2^{n\,(2n+1)}}\ \prod_{i=1}^n \mathrm{d}\lambda_i\ \left(1-\lambda_i^2\right)\ (1-\lambda_i)^2\ \Delta(\lambda_1,\dots,\lambda_n)^4\ .\end{aligned} \tag{4.14}$$

With the definitions (3.9) and (4.7), the radial element

$$R=\begin{pmatrix}\sigma^2 & \\ & \exp\left(\mathrm{i}\,\sigma^1\otimes\xi\right)\end{pmatrix} \tag{4.15}$$

satisfies the required condition (3.11) and also $R^\top=R^{-1}=R^\dagger$. The angular matrices $V\in USp(2n+2)\times USp(2n)$ are decomposed as in (3.12) with $V_+\in USp(2n+2)$ and $V_-\in USp(2n)$, so that (4.9) is obeyed. The σ-model action can be worked out analogously to (3.20) and (4.10), giving

$$\begin{aligned}Z_1^{(2n+1)}(\mu_0,\mu_1,\dots,\mu_n) &= \prod_{i=1}^n\ \int_{-1}^{1}\mathrm{d}\lambda_i\ \left(1-\lambda_i^2\right)\ (1-\lambda_i)^2\ \Delta(\lambda_1,\dots,\lambda_n)^4\\ &\quad\times\Omega_1^{(n+1)}(1,\lambda_1,\dots,\lambda_n;\mu_0,\mu_1,\dots,\mu_n)\ \Omega_1^{(n)}(\lambda_1,\dots,\lambda_n;\mu_1,\dots,\mu_n)\\ &\quad+\ (-1)^{N_{\not{D}}}\ \left\{\begin{smallmatrix}\mu_0\to-\mu_0\\ \mu_i\to-\mu_i\end{smallmatrix}\right\}\ .\end{aligned} \tag{4.16}$$

For $n = 0$ this expression is understood as being equal to $\Omega_1^{(1)}(1;\mu_0) + (-1)^{N_{\not D}}\,\Omega_1^{(1)}(1;-\mu_0)$. In [18] an expression for the finite volume partition function in this case is derived from random matrix theory in the case of a vanishing unpaired quark mass $\mu_0 = 0$ (so that the matrix model measure is positive and the partition function vanishes when $N_{\not D}$ is odd). In the next subsection we derive a formula for the equal mass limit of (4.16) without this restriction.

4.2 Sum Rules

$N_f = 2n$

As in the previous section, to work out the massive spectral sum rules (2.13) in the even-flavoured case at $\beta = 1$ it is convenient to work directly with the partition function (4.11) for equal quark masses $\mu_i = \mu \;\; \forall i = 1,\dots,n$, which is determined entirely by the n radial Goldstone boson degrees of freedom and is simply given by

$$Z_1^{(2n)}(\mu,\dots,\mu) = \prod_{i=1}^{n}\int_{-1}^{1}\mathrm{d}\lambda_i\;\left(1-\lambda_i^2\right)\;\mathrm{e}^{\,4\mu\,\lambda_i}\;\Delta(\lambda_1,\dots,\lambda_n)^4\;. \tag{4.17}$$

Because of the original fermion determinant required in (2.12) and the doubling of the flavour symmetry in this case, we can anticipate the appearence of a Pfaffian expression for the finite volume partition function, in contrast to the determinant formula of the previous section. For this, we write the power of the Vandermonde determinant in (4.17) as the determinant of a $2n \times 2n$ matrix given by [32, 40]

$$\Delta(\lambda_1,\dots,\lambda_n)^4 = \det\begin{bmatrix}\lambda_i^{j-1}\\ (j-1)\,\lambda_i^{j-2}\end{bmatrix}_{\substack{i=1,\dots,n\\ j=1,\dots,2n}}\;. \tag{4.18}$$

Expanding the determinant into a sum over permutations $\hat{w} \in S_{2n}$, and using the permutation symmetry of the integration measure and domain in (4.17) to appropriately rearrange rows, we may bring the equal mass partition function into the form

$$\begin{aligned}Z_1^{(2n)}(\mu,\dots,\mu) \;&=\; \sum_{\hat{w}\in S_{2n}}\mathrm{sgn}\,(\hat{w})\;\prod_{i=1}^{n}\int_{-1}^{1}\mathrm{d}\lambda_i\;\left(1-\lambda_i^2\right)\;\mathrm{e}^{\,4\mu\,\lambda_i}\\ &\qquad\times\Big(\hat{w}(2i)-1\Big)\;\lambda_i^{\hat{w}(2i)+\hat{w}(2i-1)-3}\;.\end{aligned} \tag{4.19}$$

Let $S_{2n}^{>}$ denote the subgroup of S_{2n} consisting of "increasing" permutations $\hat{w}$, i.e. those obeying $\hat{w}(2i) > \hat{w}(2i-1) \;\; \forall i = 1,\dots,n$. Then the expression (4.19) can be written as

$$\begin{aligned}Z_1^{(2n)}(\mu,\dots,\mu) \;&=\; \sum_{\hat{w}\in S_{2n}^{>}}\mathrm{sgn}\,(\hat{w})\;\prod_{i=1}^{n}\int_{-1}^{1}\mathrm{d}\lambda_i\;\left(1-\lambda_i^2\right)\;\mathrm{e}^{\,4\mu\,\lambda_i}\\ &\qquad\times\Big(\hat{w}(2i)-\hat{w}(2i-1)\Big)\;\lambda_i^{\hat{w}(2i)+\hat{w}(2i-1)-3}\;.\end{aligned} \tag{4.20}$$

The expression (4.20) now has the standard form of a Pfaffian [32] and we may thereby write it as

$$Z_1^{(2n)}(\mu,\dots,\mu) = \mathrm{Pfaff}\;\boldsymbol{A}_{2n}(4\mu)\;, \tag{4.21}$$

where $\boldsymbol{A}_{2n}(x)$ is the $2n \times 2n$ antisymmetric matrix with elements

$$
\begin{aligned}
\boldsymbol{A}_{2n}(x)_{ij} &= (i-j) \int_{-1}^{1} \mathrm{d}\lambda \ \left(1-\lambda^2\right) \ \lambda^{i+j-3} \ \mathrm{e}^{\,x\,\lambda} \\
&= 2(i-j)\left(\mathsf{K}^{(i+j-3)}(x) - \mathsf{K}^{(i+j-1)}(x)\right) .
\end{aligned}
\tag{4.22}
$$

These matrix elements can be alternatively expressed as

$$
\boldsymbol{A}_{2n}(x)_{ij} = 2(i-j) \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{i+j-3} \int_{-1}^{1} \mathrm{d}s \ s \int_{-1}^{s} \mathrm{d}\lambda \ \mathrm{e}^{\,x\,\lambda} = (i-j) \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{i+j-3} \frac{\mathsf{K}'(x)}{x} . \tag{4.23}
$$

The substitution of (4.23) into (4.21) can be straightforwardly shown to coincide with the equal mass limit of the finite volume partition function (4.13) obtained from the orthogonal ensemble of random matrix theory.

Proceeding as in Section 3.3 to rewrite derivatives of the spectral sine-kernel in terms of the functions (3.32), the equal mass partition function in the case of an even number of pseudo-real quarks acquires the Pfaffian form

$$
\boxed{Z_1^{(2n)}(\mu,\dots,\mu) = \mathrm{Pfaff}\left[(i-j)\left(\boldsymbol{K}_{i+j-3}(4\mu) - \boldsymbol{K}_{i+j-1}(4\mu)\right)\right]_{i,j=1,\dots,2n}} \tag{4.24}
$$

It can be computed explicitly by using the Laplace expansion of the Pfaffian [32]

$$
\mathrm{Pfaff}\ \boldsymbol{A} = \sum_{j=1}^{2n} (-1)^{i+j} \ (\boldsymbol{A})_{ij} \ \mathrm{Pfaff}\ \boldsymbol{A}_{ji,ji} \tag{4.25}
$$

for any fixed $i = 1,\dots,2n$, where the Pfaffian minor $\boldsymbol{A}_{ji,ji}$ is obtained from the antisymmetric $2n \times 2n$ matrix $\boldsymbol{A}$ by deleting both its i-th and j-th rows and columns. The massive spectral sum rules can thus be extracted from (3.32) and (4.24) by the substitution $N_f \to 2N_f$ in (2.13) to account for the symplectic doubling of fermion degrees of freedom in this case, and for the first few even numbers of quark flavours we find

$$
\boxed{
\begin{aligned}
\left\langle \sum_{\zeta>0} \frac{1}{\zeta^2+\mu^2} \right\rangle_1 \Bigg|_{N_f=2} &= \frac{1}{4\mu\,\coth 4\mu - 1} - \frac{3}{16\mu^2} \\
\left\langle \sum_{\zeta>0} \frac{1}{\zeta^2+\mu^2} \right\rangle_1 \Bigg|_{N_f=4} &= \frac{1}{16\mu^2} \ \frac{36\mu\left(4\mu^2+5\right)\sinh 8\mu - 9\left(64\mu^2+5\right)\sinh^2 4\mu - 16\mu^2\left(256\mu^4+216\mu^2+27\right)}{9(8\mu^2+1)\sinh^2 4\mu - 36\mu\,\sinh 8\mu + 128\mu^6(16\mu^2+9)}
\end{aligned}
}
\tag{4.26}
$$

These results agree with (2.14) in the limit $\mu \to 0$ of massless quarks.

$N_f = 2n+1$

The equal mass limit $\mu_i = \mu \ \ \forall i = 0, 1, \ldots, n$ of the partition function (4.16) is given by

$$\begin{aligned} Z_1^{(2n+1)}(\mu, \ldots, \mu) &= \mathrm{e}^{2\mu} \prod_{i=1}^{n} \int_{-1}^{1} \mathrm{d}\lambda_i \ \left(1-\lambda_i^2\right) \ (1-\lambda_i)^2 \ \mathrm{e}^{4\mu\lambda_i} \ \Delta(\lambda_1, \ldots, \lambda_n)^4 \\ &\quad + \ (-1)^{N_{\not D}} \ \{\mu \to -\mu\} \ . \end{aligned} \tag{4.27}$$

The calculation is identical to that of the even-flavoured case above. The only difference is that the antisymmetric matrix (4.22) now contains an additional measure factor $(1-\lambda)^2$, which when expanded leads to the equal mass finite volume partition function in the case of an odd number of symplectic fermions in the Pfaffian form

$$\boxed{\begin{aligned} &Z_1^{(2n+1)}(\mu,\ldots,\mu) = \mathrm{e}^{2\mu} \ \mathrm{Pfaff}\Big[(i-j)\Big(\mathbf{K}_{i+j-3}(4\mu) - 2\mathbf{K}_{i+j-2}(4\mu) + 2\mathbf{K}_{i+j}(4\mu) - \mathbf{K}_{i+j+1}(4\mu)\Big)\Big]_{i,j=1,\ldots,2n} \\ &\quad + (-1)^{N_{\not D}} \ \mathrm{e}^{-2\mu} \ \mathrm{Pfaff}\Big[(i-j)\Big(\mathbf{K}_{i+j-3}(4\mu) + 2\mathbf{K}_{i+j-2}(4\mu) - 2\mathbf{K}_{i+j}(4\mu) - \mathbf{K}_{i+j+1}(4\mu)\Big)\Big]_{i,j=1,\ldots,2n} \end{aligned}} \tag{4.28}$$

This expression also covers the case $N_f = 1$ of a single fermion of mass μ, obtained by dropping the Pfaffians in (4.28), which for an even number $N_{\not D}$ of Dirac operator eigenvalues leads to the massive spectral sum rule

$$\boxed{\left\langle \sum_{\zeta>0} \frac{1}{\zeta^2 + \mu^2} \right\rangle_1 \Bigg|_{N_f=1} = \frac{\tanh 2\mu}{4\mu}} \tag{4.29}$$

Sum rules for higher numbers of flavours are also straightforward to work out, but they become increasingly complicated and will not be displayed here. In each case one finds that the massless spectral sum rule (2.14) for $\beta = 1$ is indeed obeyed. The equations (2.10), (2.11), (4.24) and (4.28) suffice to express to smallest eigenvalue distribution of the QCD$_3$ Dirac operator for $SU(2)$ colour group in a closed form for an arbitrary number of fermion flavours in the case of equal quark masses.

5 Adjoint Fermions

Finally, we consider the case of quark spinors which transform in the adjoint representation of the $SU(N_c)$ colour group for $N_c \geq 2$, which is real. As in the previous section, the adjoint representation of the Euclidean Dirac operator $\mathrm{i}\not D$ uniquely defines an anti-unitary operator $\mathfrak{C}_0$ with $[\mathfrak{C}_0, \mathrm{i}\not D] = 0$, but now $(\mathfrak{C}_0)^2 = -\mathbb{1}$ [9, 11]. The operator $\mathfrak{C}_0$ is given as a combination of charge conjugation $C = \mathrm{i}\,\sigma^2$ and complex conjugation. In this case the Dirac operator can be expressed in terms of real quaternions [11], and the condition $\mathfrak{C}_0|\psi_k\rangle = |\psi_k\rangle$ defines a basis $|\psi_k\rangle$ of Majorana spinors of the second kind. The generic unitary $U(N_f)$ flavour symmetry thereby

truncates to its real orthogonal subgroup $O(N_f)$. In matrix model language this corresponds to the symplectic ensemble $\beta = 4$. The mass and parity matrices are given exactly as in (3.1), and so the Goldstone manifolds are

$$\begin{aligned} \mathcal{G}_4(2n) &= O(2n) \,/\, O(n) \times O(n) \ , & (5.1) \\ \mathcal{G}_4(2n+1) &= O(2n+1) \,/\, O(n+1) \times O(n) & (5.2) \end{aligned}$$

of real dimensions $\dim \mathcal{G}_4(2n) = n^2$ and $\dim \mathcal{G}_4(2n+1) = n\,(n+1)$. As expected, the number of Goldstone bosons here is exactly half that of Section 3.

5.1 Partition Functions

$N_f = 2$

The "degenerate" case $N_f = 2$ corresponding to two-flavour component adjoint fermions is very special because it represents flavour symmetry breaking from $O(2)$ to a *finite* subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$. The coset space (5.1) in this case is the orbifold

$$\mathcal{G}_4(2) \ = \ O(2) \,/\, \mathbb{Z}_2 \times \mathbb{Z}_2 \ . \tag{5.3}$$

The quotient by one of the $\mathbb{Z}_2$ factors in (5.3) can be used to eliminate one of the connected components of the $O(2)$ group, thereby realizing the orbifold explicitly as a one-dimensional manifold diffeomorphic to an open interval

$$\mathcal{G}_4(2) \ = \ SO(2) \,/\, \mathbb{Z}_2 \ \cong \ (0, \pi) \ . \tag{5.4}$$

After flavour symmetry breaking, there is thus a single massless degree of freedom described in the mesoscopic scaling regime by a real scalar field $\theta \in (0, \pi)$.

To evaluate the corresponding finite volume partition function (2.6), we parametrize the elements $U \in SO(2)$ appearing in (5.4) as

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{5.5}$$

and work out the effective σ-model action to be

$$\mathrm{Re}\ \mathrm{Tr}\, \Big(M\, U\, \Gamma\, U^\top \Big) = 2\mu\, \cos 2\theta \ . \tag{5.6}$$

With the normalized Haar measure $[\mathrm{d}U] = \mathrm{d}\theta/\pi$ on $SO(2)$, the partition function (2.6) in the two flavour case is thereby found to be

$$\boxed{Z_4^{(2)}(\mu) = I_0(2\mu)} \tag{5.7}$$

where generally

$$I_m(x) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{2\pi}\ \ \mathrm{e}^{\,x\,\cos\phi + \mathrm{i}\,m\,\phi} \tag{5.8}$$

is the modified Bessel function of the first kind of integer order m.

$N_f = 2n$

Let us now turn to the case of generic even integer values $N_f = 2n$ with $n > 1$. The restricted root lattice of the symmetric space (5.1) is the root space D_n of the special orthogonal group $SO(2n)$ (see Table 1), with root multiplicities $m_{\vec{e}_i \pm \vec{e}_j} = 1$ for $i, j = 1, \dots, n$, $i < j$ [25, 27]. The corresponding radial integration measure (2.25) on $\mathcal{G}_4(2n)$ is thus

$$\mathrm{d}\nu_4(U) = [\mathrm{d}V] \ \prod_{i=1}^{n} \mathrm{d}r_i \ \prod_{i<j} \Big| \sin(r_i - r_j) \ \sin(r_i + r_j) \Big| \ . \tag{5.9}$$

Similarly to the $n = 1$ case, the quotient in (5.1) can be used to eliminate the connected components of the orthogonal groups, at the expense of presenting the coset space as the orbifold

$$\mathcal{G}_4(2n) \ = \ SO(2n) \, / \, SO(n) \times SO(n) \times \mathbb{Z}_2 \ . \tag{5.10}$$

In this case, it is more convenient to parametrize the radial directions of this orbifold in terms of the angle variables $\theta_i = 2r_i \in [0, \pi]$, and using (3.6) we may write (5.9) as

$$\mathrm{d}\nu_4(U) = \frac{[\mathrm{d}V]}{2^{\frac{1}{2}\, n\,(n+1)}} \ \prod_{i=1}^{n} \mathrm{d}\theta_i \ \Big| \Delta(\cos\theta_1, \dots, \cos\theta_n) \Big| \ . \tag{5.11}$$

To ensure positivity of this measure when the absolute value signs are dropped, we will restrict the integration region for the θ_i's to the domain

$$\mathcal{D}_n^\theta = \Big\{ (\theta_1, \dots, \theta_n) \ \Big| \ 0 \le \theta_1 \le \theta_2 \le \ldots \le \theta_n \le \pi \Big\} \ . \tag{5.12}$$

This can always be achieved without complications for the integration of symmetric functions.

The particular details of the radial decomposition here are identical to those of Section 3.1, and we thereby arrive at the non-linear σ-model action

$$\begin{aligned} \mathrm{Re} \ \mathrm{Tr} \ \Big(M \, U \, \Gamma \, U^\top \Big) \ &= \ \mathrm{Tr} \left[\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} V_+ \begin{pmatrix} \cos\theta_1 & & \\ & \ddots & \\ & & \cos\theta_n \end{pmatrix} V_+^{-1} \right] \\ & \quad + \ \mathrm{Tr} \left[\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} V_- \begin{pmatrix} \cos\theta_1 & & \\ & \ddots & \\ & & \cos\theta_n \end{pmatrix} V_-^{-1} \right] \end{aligned} \tag{5.13}$$

where now $V_\pm \in O(n)$. The finite volume partition function is thus given by

$$Z_4^{(2n)}(\mu_1, \dots, \mu_n) = \int\limits_{\mathcal{D}_n^\theta} \ \prod_{i=1}^{n} \mathrm{d}\theta_i \ \Delta(\cos\theta_1, \dots, \cos\theta_n) \ \Omega_4^{(n)}(\cos\theta_1, \dots, \cos\theta_n; \mu_1, \dots, \mu_n)^2 \tag{5.14}$$

where

$$\begin{aligned} & \Omega_4^{(n)}(x_1, \dots, x_n; \mu_1, \dots, \mu_n) \\ & \qquad = \ \int\limits_{SO(n)} [\mathrm{d}V] \ \exp \ \mathrm{Tr} \left[\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} V \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} V^{-1} \right] \ . \end{aligned} \tag{5.15}$$

As in the previous section, the integral (5.15) cannot be evaluated using the Harish-Chandra formula (2.26) because the Lie algebra $\mathfrak{so}(n)$ consists of antisymmetric $n \times n$ matrices. The first two members of this sequence of angular integrals are easily evaluated explicitly to be

$$\begin{aligned} \Omega_4^{(1)}(x;\mu) &= \mathrm{e}^{\mu\, x} \ , \\ \Omega_4^{(2)}(x_1,x_2;\mu_1,\mu_2) &= 2\ \mathrm{e}^{\frac12\,(\mu_1+\mu_2)\,(x_1+x_2)}\ I_0\left(\tfrac12\,(\mu_1-\mu_2)\,(x_1-x_2)\right)\ . \end{aligned} \tag{5.16}$$

For $n=3$ the integral (5.15) can be computed by means of the Harish-Chandra formula for $SU(2) \cong SO(3)$.

In [18] the finite volume partition function in this instance is computed using a skew-orthogonal polynomial formalism in the symplectic ensemble of random matrix theory. For $n = 2\ell$ with $\ell \geq 1$ it is necessary to assume that the mass eigenvalues μ_i appear in degenerate pairs, i.e. $\mu_{2a-1} = \mu_{2a}$ for each $a = 1,\dots,\ell$, in order to ensure positivity of the matrix model measure. In this case a Pfaffian expression for the partition function is obtained in terms of integrals of the spectral sine kernel (3.16) (rather than derivatives as in the previous section), or equivalently in terms of exponential integral functions $\mathrm{Ei}(x)$. In general, this restriction does not aid the field theory calculation, which should thereby capture the case of generic mass eigenvalue configurations. In the next subsection we will evaluate (5.14) in the equal mass limit and find a striking contrast between the field theory and random matrix theory expressions. As in (5.7), the field theory calculation will naturally involve modified Bessel functions of the first kind (along with sine kernel derivatives), which is reminescent of results for QCD in *four* spacetime dimensions [2, 30]. This may be a consequence of the known relations between the QCD_3 and QCD_4 partition functions in the ϵ-regime [20, 22]. Comparison of our expressions with those of [18] imply that some non-trivial simplifying identities among these Bessel functions ought to hold, but we have not been able to find any.

For $n = 2\ell+1$, in addition to pairwise degeneracy, it is necessary to assume in the random matrix theory approach that the single unpaired quark mass vanishes, again for positivity reasons. In the next subsection we derive the equal mass limit of (5.14) without this restriction. From the field theory perspective, we shall also encounter a distinction between the cases $N_f = 4\ell$ and $N_f = 4\ell+2$, and this distinction is very natural. From Table 1 we see that the Weyl group sum for the group $SO(2\ell+1)$ is the same as that of the symplectic group $Sp(2\ell)$. In contrast, the Weyl group of $SO(2\ell)$ contains the usual permutation part S_ℓ and the factor $(\mathbb{Z}_2)^{\ell-1}$ is the normal subgroup given by the possible assignments of an *even* number of sign flips to the eigenvalues. These facts, along with observations similar to those spelled out in the previous section, support the precise matching between the field theory and random matrix theory results in these instances.

$N_f = 2n+1$

The restricted root lattice of the Goldstone manifold (5.2) is the root lattice B_n of the special orthogonal group $SO(2n+1)$ (see Table 1), with multiplicities $m_{\vec e_i \pm \vec e_j} = m_{\vec e_i} = 1$ for $i,j = 1,\dots,n$, $i<j$ [25, 27]. The corresponding coset space measure (2.25) is thus

$$
\begin{aligned}
\mathrm{d}\nu_4(U) &= [\mathrm{d}V] \prod_{i=1}^{n} \mathrm{d}r_i \ \sin r_i \ \prod_{i<j} \Big| \sin(r_i - r_j) \ \sin(r_i + r_j) \Big| \\
&= [\mathrm{d}V] \prod_{i=1}^{n} \mathrm{d}\kappa_i \ \left| \Delta\left(\kappa_1^2, \dots, \kappa_n^2\right) \right| \ , \qquad (5.17)
\end{aligned}
$$

where we have defined $\kappa_i = \cos r_i \in [0,1]$ and used (3.6) along with the double angle identity. By introducing the domain

$$
\mathcal{D}_n^\kappa = \Big\{ (\kappa_1, \dots, \kappa_n) \ \Big| \ 0 \le \kappa_1 \le \kappa_2 \le \dots \le \kappa_n \le 1 \Big\} \qquad (5.18)
$$

and proceeding exactly as in Section 3.1, the finite volume partition function may then be written as

$$
\begin{aligned}
Z_4^{(2n+1)}(\mu_0, \mu_1, \dots, \mu_n) &= \int\limits_{\mathcal{D}_n^\kappa} \prod_{i=1}^{n} \mathrm{d}\kappa_i \ \Delta\left(\kappa_1^2, \dots, \kappa_n^2\right) \ \Omega_4^{(n)}\left(2\kappa_1^2 - 1, \dots, 2\kappa_n^2 - 1; \mu_1, \dots, \mu_n\right) \\
&\quad \times \Omega_4^{(n+1)}\left(1, 2\kappa_1^2 - 1, \dots, 2\kappa_n^2 - 1; \mu_0, \mu_1, \dots, \mu_n\right) \\
&\quad + \ (-1)^{N_{\not{D}}} \ \left\{ {}^{\mu_0 \to -\mu_0}_{\mu_i \to -\mu_i} \right\} \qquad (5.19)
\end{aligned}
$$

with the same conventions for the $n = 0$ case as in (4.16). The odd-flavoured cases are not covered by the random matrix theory analysis of [18]. In the next subsection we will compute (5.19) explicitly in the case of equal quark masses. We will find that, in contrast to the previous two classes, for adjoint fermions they involve fundamentally different functional structures than those of the even-flavoured cases. In this case the partition function is expressed in terms of a combination of error functions and Hermite polynomials, and appears to be more akin to the random matrix theory predictions [18, 32]. These functional forms are similiar to those which arise in partially-quenched effective field theories of QCD$_3$ [19].

5.2 Sum Rules

To compute the massive spectral sum rules (2.13), we proceed as previously to compute directly the finite volume partition functions (2.6) in the limit of equal quark masses for the various congruence classes modulo 4 of the flavour number N_f.

$N_f = 2$

For $N_f = 2$, we substitute (5.7) into (2.13) to compute

$$
\boxed{ \left\langle \sum_{\zeta > 0} \frac{1}{\zeta^2 + \mu^2} \right\rangle_4 \Bigg|_{N_f = 2} = \frac{1}{2\mu} \, \frac{I_1(2\mu)}{I_0(2\mu)} } \qquad (5.20)
$$

Once again this agrees with (2.14) in the limit $\mu \to 0$.

$N_f = 4\ell$

For $N_f = 2n$ with $n > 1$ the partition function (5.14) for equal quark masses is given by

$$Z_4^{(2n)}(\mu, \ldots, \mu) = \int_{\mathcal{D}_n^\theta} \prod_{i=1}^{n} \mathrm{d}\theta_i \ \mathrm{e}^{2\mu \cos\theta_i} \ \Delta(\cos\theta_1, \ldots, \cos\theta_n) \ . \tag{5.21}$$

Again, since the functional integration in the original QCD$_3$ partition function is over *real* Majorana fermion fields, thereby producing a square-root fermion determinant in (2.12), we can anticipate a Pfaffian expression for (5.21). With this in mind, we will proceed to evaluate the finite volume partition function using the method of integration over alternate variables [32, 40], whose details again depend crucially on the even/odd parity of the flavour rank n.

For $n = 2\ell$ with $\ell \geq 1$ we can insert the determinant representation (3.26) to write

$$Z_4^{(4\ell)}(\mu, \ldots, \mu) \ = \ \int_{\mathcal{D}_{2\ell}^\theta} \prod_{a=1}^{\ell} \mathrm{d}\theta_{2a} \ \mathrm{e}^{2\mu \cos\theta_{2a}} \int_{\theta_{2a-2}}^{\theta_{2a}} \mathrm{d}\theta_{2a-1} \ \mathrm{e}^{2\mu \cos\theta_{2a-1}} \ \det\left[\cos^{j-1}\theta_i\right]_{i,j=1,\ldots,2\ell} \tag{5.22}$$

where we have defined $\theta_0 = 0$. We formally insert the integrals into the determinant in (5.22) by using its multilinearity to write

$$Z_4^{(4\ell)}(\mu, \ldots, \mu) = \int_{\mathcal{D}_{2\ell}^\theta} \prod_{a=1}^{\ell} \mathrm{d}\theta_{2a} \ \mathrm{e}^{2\mu \cos\theta_{2a}} \ \det\left[\begin{matrix} \cos^{j-1}\theta_{2b} \\ \int_{\theta_{2b-2}}^{\theta_{2b}} \mathrm{d}\theta \ \mathrm{e}^{2\mu\cos\theta} \ \cos^{j-1}\theta \end{matrix}\right]_{\substack{b=1,\ldots,\ell \\ j=1,\ldots,2\ell}} \ . \tag{5.23}$$

Since the integrand of (5.23) is a symmetric function of the integration variables, we may extend the lower limit of integration of each variable inside the determinant down to 0. This symmetry also allows us to extend the range of each of the remaining integration variables to $[0, \pi]$. After rearranging rows in the determinant and expanding it as a sum over permutations, we arrive at the expression

$$\begin{aligned} Z_4^{(4\ell)}(\mu, \ldots, \mu) \ &= \ \sum_{\hat{w} \in S_{2\ell}} \mathrm{sgn}\,(\hat{w}) \ \prod_{a=1}^{\ell} \int_0^{\pi} \mathrm{d}\theta_{2a} \ \mathrm{e}^{2\mu\cos\theta_{2a}} \ \cos^{\hat{w}(2a)-1}\theta_{2a} \\ &\quad \times \int_0^{\theta_{2a}} \mathrm{d}\theta \ \mathrm{e}^{2\mu\cos\theta} \ \cos^{\hat{w}(2a-1)-1}\theta \end{aligned} \tag{5.24}$$

which may be written as a sum over the subgroup $S_{2\ell}^{>} \subset S_{2\ell}$ of increasing permutations given by

$$\begin{aligned} Z_4^{(4\ell)}(\mu, \ldots, \mu) \ &= \ \sum_{\hat{w} \in S_{2\ell}^{>}} \mathrm{sgn}\,(\hat{w}) \ \prod_{a=1}^{\ell} \int_0^{\pi} \mathrm{d}\theta_{2a} \ \mathrm{e}^{2\mu\cos\theta_{2a}} \int_0^{\theta_{2a}} \mathrm{d}\theta \ \mathrm{e}^{2\mu\cos\theta} \\ &\quad \times \Big(\cos^{\hat{w}(2a)-1}\theta_{2a} \ \cos^{\hat{w}(2a-1)-1}\theta - \cos^{\hat{w}(2a-1)-1}\theta_{2a} \ \cos^{\hat{w}(2a)-1}\theta\Big) \ . \end{aligned} \tag{5.25}$$

As before, the expression (5.25) now has the standard form of a Pfaffian and we may write it as

$$Z_4^{(4\ell)}(\mu,\dots,\mu) = \mathrm{Pfaff}\,\boldsymbol{B}_{2\ell}(2\mu)\ , \tag{5.26}$$

where $\boldsymbol{B}_{2\ell}(x)$ is the $2\ell\times 2\ell$ antisymmetric matrix with elements

$$\boldsymbol{B}_{2\ell}(x)_{ij} = \int_0^\pi \mathrm{d}\theta\ \mathrm{e}^{\,x\,\cos\theta}\ \cos^{i-1}\theta\ \int_0^\pi \mathrm{d}\theta'\ \mathrm{sgn}\,\big(\theta-\theta'\big)\ \ \mathrm{e}^{\,x\,\cos\theta'}\ \cos^{j-1}\theta'\ . \tag{5.27}$$

The integrals in (5.27) can be evaluated explicitly by using the Fourier series representation of the sign function

$$\mathrm{sgn}(\phi) = \frac{4}{\pi}\sum_{k=0}^\infty \frac{\sin(2k+1)\phi}{2k+1} \tag{5.28}$$

for $\phi\in[-\pi,\pi]$, and defining the sequence of functions

$$\mathsf{I}_{2k+1}(x) = \int_0^\pi \mathrm{d}\theta\ \ \mathrm{e}^{\,x\,\cos\theta+(2k+1)\,\mathrm{i}\,\theta} \tag{5.29}$$

for $k\in\mathbb{N}_0$. We may then express the matrix elements (5.27) as series in derivatives of the functions (5.29) given by

$$\boldsymbol{B}_{2\ell}(x)_{ij} = \frac{\mathrm{i}}{2\pi}\sum_{k=0}^\infty \frac{\mathsf{I}_{2k+1}^{(i-1)}(x)\,\mathsf{I}_{2k+1}^{(j-1)}(x)^* - \mathsf{I}_{2k+1}^{(j-1)}(x)\,\mathsf{I}_{2k+1}^{(i-1)}(x)^*}{2k+1}\ . \tag{5.30}$$

Let us now explicitly evaluate the derivatives of the functions (5.29). The real parts can be written in terms of the modified Bessel functions (5.8) as

$$\mathrm{Re}\ \mathsf{I}_{2k+1}^{(i-1)}(x) = \pi\, I_{2k+1}^{(i-1)}(x)\ . \tag{5.31}$$

For the imaginary parts, we use the expansion

$$\begin{aligned}\sin(2k+1)\theta &= \mathrm{Im}\,(\cos\theta+\mathrm{i}\,\sin\theta)^{2k+1}\\ &= \sin\theta\sum_{l=0}^k \binom{2k+1}{2l+1}\sum_{a=0}^l(-1)^a\binom{l}{a}\ \cos^{2k-2a}\theta\end{aligned} \tag{5.32}$$

along with the change of integration variable $\lambda=\cos\theta\in[-1,1]$ in (5.29) to evaluate them in terms of the sine kernel derivatives (3.27) as

$$\mathrm{Im}\ \mathsf{I}_{2k+1}^{(i-1)}(x) = 2\sum_{l=0}^k\binom{2k+1}{2l+1}\sum_{a=0}^l(-1)^a\binom{l}{a}\ \mathsf{K}^{(2k-2a+i-1)}(x)\ . \tag{5.33}$$

The sums over l and a in (5.33) can be interchanged, after which the sum over $p=l-a$ for fixed a can be evaluated explicitly by using the combinatorial identity

$$\sum_{p=0}^{k-a}\binom{2k+1}{2p+2a+1}\binom{p+a}{a} = 4^{k-a}\binom{2k-a}{a} \tag{5.34}$$

that is straightforwardly proven by induction. Substituting in the explicit functional forms (3.32) for the sine kernel derivatives, and plugging (5.31) and (5.33) into (5.30) then enables us to write the equal mass partition function (5.26) explicitly as

$$\boxed{Z_4^{(4\ell)}(\mu,\ldots,\mu) = \mathrm{Pfaff}\left[\sum_{k=0}^{\infty} \frac{I_{2k+1}^{(i-1)}(2\mu)\,\boldsymbol{I}_{2k;j-1}(2\mu) - I_{2k+1}^{(j-1)}(2\mu)\,\boldsymbol{I}_{2k;i-1}(2\mu)}{2k+1}\right]_{i,j=1,\ldots,2\ell}} \tag{5.35}$$

where

$$\begin{aligned} \boldsymbol{I}_{k;i}(x) &= \frac{1}{2}\sum_{b=0}^{k}\sum_{a=0}^{\lfloor\frac{k+i-b}{2}\rfloor} \frac{(-1)^a\,2^{k-2a}\,(k-2a+i)!}{(k-2a-b+i)!}\binom{k-a}{a}\left(-\frac{1}{x}\right)^{b+1} \\ &\quad\times\left[\Big(1-(-1)^b\Big)\Big(\sinh x-\cosh x\Big)-2\sinh x\right] \ . \end{aligned} \tag{5.36}$$

From (5.35) one may now extract the massive spectral sum rules (2.13) by using the derivative identities

$$\frac{\mathrm{d}I_m(x)}{\mathrm{d}x} = I_{m+1}(x) + \frac{m}{x}\,I_m(x) \ , \quad \frac{\mathrm{d}\boldsymbol{I}_{k;i}(x)}{\mathrm{d}x} = \boldsymbol{I}_{k;i+1}(x) \ . \tag{5.37}$$

For instance, for four flavours of adjoint fermions one finds

$$\boxed{\left\langle\sum_{\zeta>0}\frac{1}{\zeta^2+\mu^2}\right\rangle_4\Bigg|_{N_f=4} = \frac{\sum\limits_{k=0}^{\infty}\frac{I_{2k+1}(2\mu)\left(\mathbf{I}_{k;2}(2\mu)-\frac{4k^2-1}{4\mu^2}\,\mathbf{I}_{k;0}(2\mu)\right)-\mathbf{I}_{k;0}(2\mu)\left(I_{2k+3}(2\mu)+\frac{4k+3}{2\mu}\,I_{2k+2}(2\mu)\right)}{2k+1}}{4\mu\sum\limits_{k=0}^{\infty}\frac{I_{2k+1}(2\mu)\left(\mathbf{I}_{k;1}(2\mu)-\frac{2k+1}{2\mu}\,\mathbf{I}_{k;0}(2\mu)\right)-I_{2k+2}(2\mu)\,\mathbf{I}_{k;0}(2\mu)}{2k+1}}} \tag{5.38}$$

A numerical evaluation of (5.38) in the limit $\mu \to 0$ establishes precise agreement with (2.14) in this case.

$N_f = 4\ell + 2$

Let us now consider the case $N_f = 2n$ with generic odd flavour rank $n = 2\ell + 1$, $\ell \geq 1$. The calculation proceeds analogously to that of the even rank case, except that now one needs to keep careful track of the single unpaired integration. Starting from (5.21) we derive the determinant formula

$$
\begin{aligned}
Z_4^{(4\ell+2)}(\mu,\dots,\mu) &= \int\limits_{\mathcal{D}^\theta_{2\ell+1}} \prod_{a=1}^{\ell} \mathrm{d}\theta_{2a}\ \mathrm{e}^{\,2\mu\cos\theta_{2a}} \int\limits_{\theta_{2a-2}}^{\theta_{2a}} \mathrm{d}\theta_{2a-1}\ \mathrm{e}^{\,2\mu\cos\theta_{2a-1}} \\
&\qquad \times \det\left[\cos^{j-1}\theta_i\right]_{i,j=1,\dots,2\ell+1} \int\limits_{\theta_{2\ell}}^{\pi} \mathrm{d}\theta_{2\ell+1}\ \mathrm{e}^{\,2\mu\cos\theta_{2\ell+1}} \\
&= \int\limits_{\mathcal{D}^\theta_{2\ell+1}} \prod_{a=1}^{\ell} \mathrm{d}\theta_{2a}\ \mathrm{e}^{\,2\mu\cos\theta_{2a}}\ \det\begin{bmatrix} \cos^{j-1}\theta_{2b} \\ \int_{\theta_{2b-2}}^{\theta_{2b}} \mathrm{d}\theta\ \mathrm{e}^{\,2\mu\cos\theta}\ \cos^{j-1}\theta \\ \int_{\theta_{2\ell}}^{\pi} \mathrm{d}\phi\ \mathrm{e}^{\,2\mu\cos\phi}\ \cos^{j-1}\phi \end{bmatrix}_{\substack{b=1,\dots,\ell \\ j=1,\dots,2\ell+1}} .
\end{aligned}
\tag{5.39}
$$

Using symmetry of the integrand in (5.39) to extend the integration ranges as before, expansion of the determinant into a sum over permutations yields

$$
\begin{aligned}
Z_4^{(4\ell+2)}(\mu,\dots,\mu) &= \sum_{\hat{w}\in S_{2\ell+1}} \mathrm{sgn}\,(\hat{w}) \prod_{a=1}^{\ell} \int\limits_0^{\pi} \mathrm{d}\theta_{2a}\ \mathrm{e}^{\,2\mu\cos\theta_{2a}}\ \cos^{\hat{w}(2a)-1}\theta_{2a} \\
&\qquad \times \int\limits_0^{\theta_{2a}} \mathrm{d}\theta\ \mathrm{e}^{\,2\mu\cos\theta}\ \cos^{\hat{w}(2a-1)-1}\theta \int\limits_0^{\pi} \mathrm{d}\phi\ \mathrm{e}^{\,2\mu\cos\phi}\ \cos^{\hat{w}(2\ell+1)-1}\phi \\
&= \sum_{\hat{w}\in S^>_{2\ell+1}} \mathrm{sgn}\,(\hat{w}) \prod_{a=1}^{\ell} \int\limits_0^{\pi} \mathrm{d}\theta_{2a}\ \mathrm{e}^{\,2\mu\cos\theta_{2a}} \int\limits_0^{\theta_{2a}} \mathrm{d}\theta\ \mathrm{e}^{\,2\mu\cos\theta} \\
&\qquad \times \Big(\cos^{\hat{w}(2a)-1}\theta_{2a}\ \cos^{\hat{w}(2a-1)-1}\theta - \cos^{\hat{w}(2a-1)-1}\theta_{2a}\ \cos^{\hat{w}(2a)-1}\theta\Big) \\
&\qquad \times \int\limits_0^{\pi} \mathrm{d}\phi\ \mathrm{e}^{\,2\mu\cos\phi}\ \cos^{\hat{w}(2\ell+1)-1}\phi
\end{aligned}
\tag{5.40}
$$

which again has the standard form of a Pfaffian of dimension $(2\ell+2)\times(2\ell+2)$. The last integral in (5.40) can be expressed in terms of derivatives of the modified Bessel function $I_0(2\mu)$. With the same functions (5.36), we thereby find the equal mass finite volume partition function in the explicit form

$$
\boxed{
\begin{aligned}
&Z_4^{(4\ell+2)}(\mu,\dots,\mu) \\
&\qquad = \mathrm{Pfaff}\begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{I_{2k+1}^{(i-1)}(2\mu)\,\mathbf{I}_{k;j-1}(2\mu) - I_{2k+1}^{(j-1)}(2\mu)\,\mathbf{I}_{k;i-1}(2\mu)}{2k+1} & I_0^{(i-1)}(2\mu) \\ -I_0^{(j-1)}(2\mu) & 0 \end{bmatrix}_{i,j=1,\dots,2\ell+1}
\end{aligned}
}
\tag{5.41}
$$

The (rather cumbersome) sum rules may now be extracted from (5.41) exactly as described above for the even rank case.

$N_f = 4\ell + 1$

For odd numbers of flavours $N_f = 2n+1$ the equal mass limit of the finite volume partition function (5.19) is given by

$$Z_4^{(2n+1)}(\mu,\dots,\mu) = \mathrm{e}^{-(2n-1)\,\mu} \int\limits_{\mathcal{D}_n^\kappa} \prod_{i=1}^n \mathrm{d}\kappa_i \ \mathrm{e}^{4\mu\,\kappa_i^2}\ \Delta\left(\kappa_1^2,\dots,\kappa_n^2\right) \quad + \quad (-1)^{N_{\not p}}\ \{\mu\to-\mu\}\ . \tag{5.42}$$

The calculation of the integrals in (5.42) is identical to that of (5.21), with the obvious modification of the integration domain and the replacements $\cos\theta_i \to 2\kappa_i^2$ everywhere in the previous formulas. For $n = 2\ell$, one may in this way bring (5.42) into the Pfaffian form

$$Z_4^{(4\ell+1)}(\mu,\dots,\mu) = \mathrm{e}^{-(4\ell-1)\,\mu}\ \mathrm{Pfaff}\ \boldsymbol{E}_{2\ell}(4\mu) + (-1)^{N_{\not p}}\ \mathrm{e}^{(4\ell-1)\,\mu}\ \mathrm{Pfaff}\ \boldsymbol{E}_{2\ell}(-4\mu) \tag{5.43}$$

where $\boldsymbol{E}_{2\ell}(x)$ is the $2\ell\times2\ell$ antisymmetric matrix defined by the elements

$$\boldsymbol{E}_{2\ell}(x)_{ij} = \int\limits_0^1 \mathrm{d}\kappa\ \mathrm{e}^{x\,\kappa^2}\ \left(\kappa^2\right)^{i-1} \int\limits_0^1 \mathrm{d}\kappa'\ \mathrm{sgn}\left(\kappa-\kappa'\right)\ \mathrm{e}^{x\,\kappa'^2}\ \left(\kappa'^2\right)^{j-1}\ . \tag{5.44}$$

In this case it is more convenient to use the Fourier integral representation of the sign function given by

$$\mathrm{sgn}(y) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}q\ \frac{\sin q\,y}{q+\mathrm{i}\,\varepsilon} \tag{5.45}$$

for $y\in\mathbb{R}$, where the infinitesimal parameter $\varepsilon\to0^+$ regulates the ambiguity in $\mathrm{sgn}(y)$ at $y=0$. Then the matrix elements (5.44) can be written in terms of the functions

$$\Xi_q(x) = \int\limits_0^1 \mathrm{d}\kappa\ \mathrm{e}^{x\,\kappa^2+\mathrm{i}\,q\,\kappa} \tag{5.46}$$

which may be integrated explicitly in terms of error functions.

In this we may write the finite volume partition function (5.43) explicitly as

$$\boxed{\begin{aligned} &Z_4^{(4\ell+1)}(\mu,\dots,\mu) \\ &\quad = \mathrm{e}^{-(4\ell-1)\,\mu}\ \mathrm{Pfaff}\left[\frac{\mathrm{i}}{4\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}q\ \frac{\Xi_q^{(i-1)}(4\mu)\,\Xi_q^{(j-1)}(4\mu)^*-\Xi_q^{(j-1)}(4\mu)\,\Xi_q^{(i-1)}(4\mu)^*}{q+\mathrm{i}\,\varepsilon}\right]_{i,j=1,\dots,2\ell} \\ &\qquad + \ (-1)^{N_{\not p}}\ \{\mu\to-\mu\} \end{aligned}} \tag{5.47}$$

where

$$\Xi_q(x) = \frac{1}{2\mathrm{i}}\sqrt{\frac{\pi}{x}}\ \mathrm{e}^{\frac{q^2}{2x}}\ \left[\mathrm{erf}\left(\frac{q}{2\sqrt{x}}\right) - \mathrm{erf}\left(\frac{q-2\,\mathrm{i}\,x}{2\sqrt{x}}\right)\right]\ . \tag{5.48}$$

The matrix elements in (5.47) can be made somewhat more explicit by using the derivative identity

$$\frac{\mathrm{d}^k \operatorname{erf}(x)}{\mathrm{d}x^k} = (-1)^{k-1} \, \frac{2}{\sqrt{\pi}} \ \mathrm{e}^{-x^2} \ H_{k-1}(x) \ , \tag{5.49}$$

where $H_m(x)$ are the Hermite polynomials. As before, this expression includes the one-flavour case $N_f = 1$ ($\ell = 0$), which is obtained from (5.47) by omitting the Pfaffians. These relations can also be used to now extract the massive spectral sum rules in these instances.

$N_f = 4\ell + 3$

The partition function (5.42) for $n = 2\ell + 1$ is evaluated identically as in (5.40) with the obvious change of integration domain and the substitutions $\cos\theta_i \to 2\kappa_i^2$ as above. The single unpaired integration in this case can be expressed in terms of derivatives of the function

$$\Xi_0(x) = \tfrac{1}{2\,\mathrm{i}} \, \sqrt{\tfrac{\pi}{x}} \, \operatorname{erf}\left(\mathrm{i}\sqrt{x}\right) \ , \tag{5.50}$$

and we can thereby write the equal mass finite volume partition function as the $(2\ell+2)\times(2\ell+2)$ Pfaffian formula

$$\boxed{\begin{aligned} & Z_4^{(4\ell+3)}(\mu,\dots,\mu) \\ & \quad = \mathrm{e}^{-(4\ell+1)\,\mu} \ \mathrm{Pfaff}\begin{bmatrix} \frac{\mathrm{i}}{4\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}q \ \frac{\Xi_q^{(i-1)}(4\mu)\,\Xi_q^{(j-1)}(4\mu)^* - \Xi_q^{(j-1)}(4\mu)\,\Xi_q^{(i-1)}(4\mu)^*}{q + \mathrm{i}\,\varepsilon} & \Xi_0^{(i-1)}(4\mu) \\ -\Xi_0^{(j-1)}(4\mu) & 0 \end{bmatrix}_{i,j=1,\dots,2\ell+1} \\ & \qquad + \ (-1)^{N_{\not D}} \ \left\{\mu \to -\mu\right\} \end{aligned}} \tag{5.51}$$

This expression encompasses the case of three flavours ($\ell = 0$) by omitting the $(2\ell+1)\times(2\ell+1)$ blocks in (5.51). The Goldstone manifold (5.1) in this case is diffeomorphic to a disk, $\mathcal{G}_4(3) \cong SO(3)/SO(2) \times \mathbb{Z}_2 \cong S^2/\mathbb{Z}_2 \cong D^2$, and the partition function is equal to the function (5.50) evaluated at $x = 4\mu$. This simple evaluation follows of course from the explicit forms of the angular integrals (5.16) in this case. The corresponding massive spectral sum rule (2.13) is given by

$$\boxed{\left\langle \sum_{\zeta>0} \frac{1}{\zeta^2 + \mu^2} \right\rangle_4 \Bigg|_{N_f=3} = \frac{4\sqrt{\mu} \ \mathrm{e}^{4\mu} - \sqrt{\pi} \operatorname{erf}\left(2\,\mathrm{i}\sqrt{\mu}\right)}{12\sqrt{\pi} \ \mu^2 \operatorname{erf}\left(2\,\mathrm{i}\sqrt{\mu}\right)}} \tag{5.52}$$

In addition to the sum rules, the expressions (2.8), (5.7), (5.35), (5.41), (5.47) and (5.51) suffice to determine the spectral k-point correlation functions in closed form for an arbitrary number of flavours N_f in the limit where the collection of equal quark mass and (imaginary) Dirac operator eigenvalues is completely degenerate. In particular, they may be used to determine the correlators at the spectral origin for massless quarks, as well as the smallest eigenvalue distribution functions (2.10,2.11).

Acknowledgments

The author is grateful to G. Akemann, P. Damgaard, D. Johnston and G. Vernizzi for helpful discussions and correspondence. This work was supported in part by a PPARC Advanced Fellowship, by PPARC Grant PPA/G/S/2002/00478, and by the EU-RTN Network Grant MRTN-CT-2004-005104.

References

[1] J. Gasser and H. Leutwyler, Phys. Lett. **B188** (1987) 477;
P.H. Damgaard, Nucl. Phys. Proc. Suppl. **128** (2004) 47 [hep-lat/0310037].

[2] H. Leutwyler and A. Smilga, Phys. Rev. **D46** (1992) 5607.

[3] C. Bernard and M.F.L. Golterman, Phys. Rev. **D46** (1992) 853 [hep-lat/9204007]; **D49** (1994) 486 [hep-lat/9306005];
M.F.L. Golterman, Acta Phys. Pol. **B25** (1994) 1731 [hep-lat/9411005].

[4] L. Giusti, C. Hoelbing, M. Lüscher and H. Wittig, Comp. Phys. Commun. **153** (2003) 31 [hep-lat/0212012];
L. Giusti, M. Lüscher, P. Weisz and H. Wittig, J. High Energy Phys. **0311** (2003) 023 [hep-lat/0309189].

[5] E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. **A560** (1993) 306 [hep-th/9212088];
J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70** (1993) 3852 [hep-th/9303012].

[6] G. Akemann, P.H. Damgaard, U. Magnea and S.M. Nishigaki, Nucl. Phys. **B487** (1997) 721 [hep-th/9609174].

[7] R.D. Pisarski, Phys. Rev. **D29** (1984) 2423.

[8] J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **73** (1994) 2288 [hep-th/9405005].

[9] M.A. Halasz and J.J.M. Verbaarschot, Phys. Rev. **D52** (1995) 2563 [hep-th/9502096].

[10] U. Magnea, Phys. Rev. **D61** (2000) 056005 [hep-th/9907096].

[11] U. Magnea, Phys. Rev. **D62** (2000) 016005 [hep-th/9912207].

[12] T. Nagao and K. Slevin, J. Math. Phys. **34** (1993) 2075.

[13] P.H. Damgaard, U.M. Heller, A. Krasnitz and T. Madsen, Phys. Lett. **B440** (1998) 129 [hep-th/9803012].

[14] P.H. Damgaard and S.M. Nishigaki, Phys. Rev. **D57** (1998) 5299 [hep-th/9711096].

[15] G. Akemann and P.H. Damgaard, Nucl. Phys. **B598** (1998) 411 [hep-th/9801133]; Phys. Lett. **B432** (1998) 390 [hep-th/9802174].

[16] J. Christiansen, Nucl. Phys. **B547** (1999) 329 [hep-th/9809194].

[17] C. Hilmoine and R. Niclasen, Phys. Rev. **D62** (2000) 096013 [hep-th/0004081].

[18] T. Nagao and S.M. Nishigaki, Phys. Rev. **D63** (2001) 045011 [hep-th/0005077].

[19] R.J. Szabo, Nucl. Phys. **B598** (2001) 309 [hep-th/0009237].

[20] G. Akemann, D. Dalmazi, P.H. Damgaard and J.J.M. Verbaarschot, Nucl. Phys. **B601** (2001) 77 [hep-th/0011072].

[21] T. Andersson, P.H. Damgaard and K. Splittorff, Nucl. Phys. **B707** (2005) 509 [hep-th/0410163].

[22] G. Akemann and P.H. Damgaard, Nucl. Phys. **B576** (2000) 597 [hep-th/9910190].

[23] H.W. Braden, A. Mironov and A. Morozov, Phys. Lett. **B514** (2001) 293 [hep-th/0105169].

[24] M.R. Zirnbauer, J. Math. Phys. **37** (1996) 4986 [math-ph/9808012].

[25] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces* (Academic Press, New York, 1978).

[26] S. Helgason, *Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators and Spherical Functions* (Academic Press, New York, 1984).

[27] M. Caselle and U. Magnea, Phys. Rept. **394** (2004) 41 [cond-mat/0304363].

[28] P.H. Damgaard, Phys. Lett. **B425** (1998) 151 [hep-th/9711047].

[29] S.M. Nishigaki, P.H. Damgaard and T. Wettig, Phys. Rev. **D58** (1998) 087704 [hep-th/9803007];
P.H. Damgaard and S.M. Nishigaki, Phys. Rev. **D63** (2001) 045012 [hep-th/0006111].

[30] A. Smilga and J.J.M. Verbaarschot, Phys. Rev. **D51** (1995) 829 [hep-th/9404031];
A.D. Jackson, M.K. Şener and J.J.M. Verbaarschot, Phys. Lett. **B387** (1996) 355 [hep-th/9605183];
T. Guhr and T. Wettig, Nucl. Phys. **B506** (1997) 589 [hep-th/9704055];
T. Nagao and S.M. Nishigaki, Phys. Rev. **D62** (2000) 065006 [hep-th/0001137].

[31] A.N. Redlich, Phys. Rev. Lett. **52** (1984) 18.

[32] M.L. Mehta, *Random Matrices* (Academic Press, San Diego, 1991).

[33] U. Magnea, Phys. Rev. **D64** (2001) 018902 [hep-th/0009208].

[34] M.R. Zirnbauer and F.D.M. Haldane, Phys. Rev. **B52** (1995) 8729 [cond-mat/9504108].

[35] Harish-Chandra, Am. J. Math. **79** (1957) 87.

[36] R.J. Szabo, *Equivariant Cohomology and Localization of Path Integrals* (Springer-Verlag, Berlin-Heidelberg, 2000).

[37] C. Itzykson and J.-B. Zuber, J. Math. Phys. **21** (1980) 411.

[38] A.B. Balentekin and P. Cassak, J. Math. Phys. **43** (2002) 604 [hep-th/0108130].

[39] B. Schlittgen and T. Wettig, J. Phys. **A36** (2003) 3195 [math-ph/0209030].

[40] T. Nagao and P.J. Forrester, Nucl. Phys. **B509** (1998) 561.